Linear Algebra Review

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physics: $\langle x | y \rangle \longrightarrow |x\rangle\langle y|$ "ketbra"

Inner Product $\longleftrightarrow$ "dot" product

$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n = x^T y$

$$\left.\begin{array}{l} \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \\ \langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \end{array}\right\} \text{linearity}$$

$\langle x, y \rangle = \langle y, x \rangle$ reflexivity

$\langle x, x \rangle \geq 0$ positivity

$\langle x, x \rangle = 0 \Longleftrightarrow x = 0$ "definite"

$f(y) = \langle x, y \rangle$ linear function $\mathbb{R}^n \to \mathbb{R}$

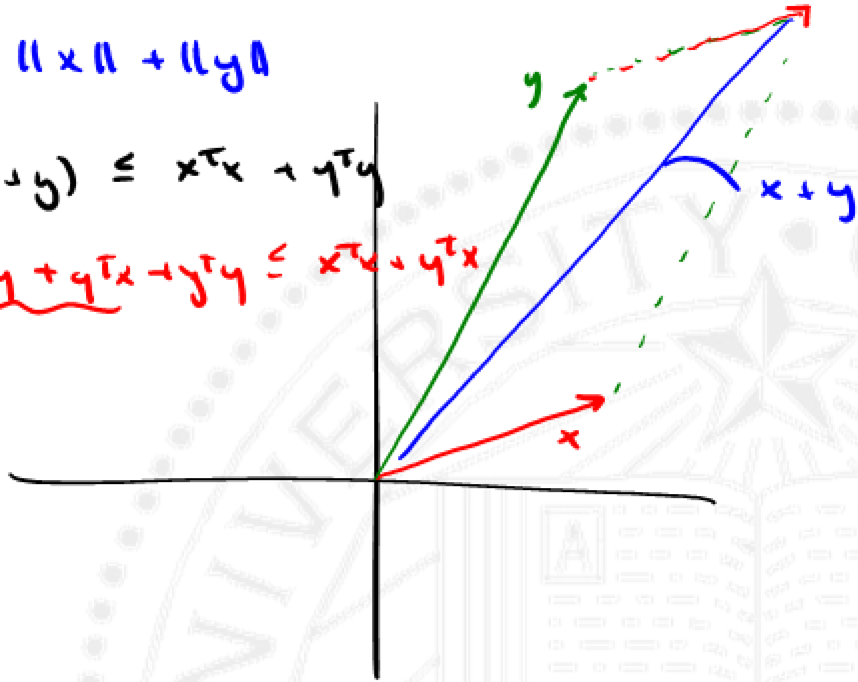
$f(y) = \underbrace{x^T}_{\in \mathbb{R}^{1\times n}} y$

Triangle Inequality

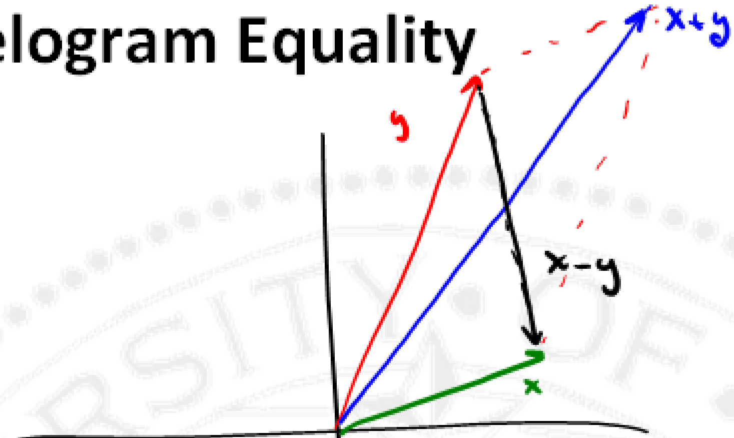
$$\|x+y\| \leq \|x\| + \|y\|$$

$$(x+y)^T(x+y) \leq x^Tx + y^Ty$$

$$x^Tx + \underbrace{x^Ty + y^Tx} + y^Ty \leq x^Tx + y^Tx$$



Parallelogram Equality



$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$x^Tx + x^Ty + y^Tx + y^Ty = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$

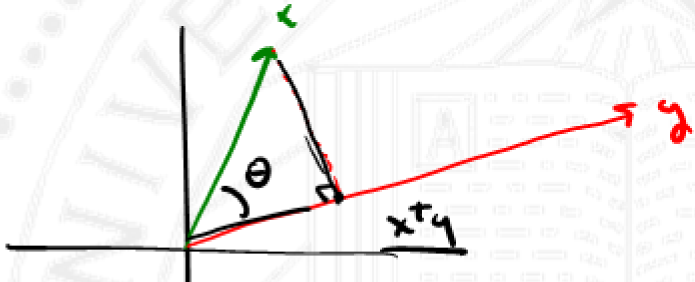
$$x^Tx - x^Ty - y^Tx + y^Ty = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$$

Cauchy-Schwartz Inequality

$x, y \in \mathbb{R}^n \qquad |x^T y| \le \|x\| \|y\|$

angle between vectors in $\mathbb{R}^n$:

$$\theta \triangleq \angle(x,y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

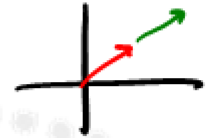


$$x^T y = \|x\| \|y\| \cos\theta$$

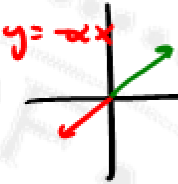
Special cases of Cauchy-Schwartz

$y = \alpha x \quad \alpha \geq 0$

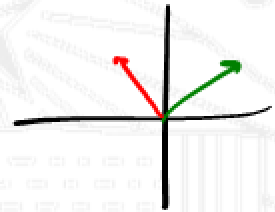
x, y — aligned $\quad \theta = \phi \quad x^T y = \|x\| \cdot \|y\|$



x, y — opposed $\quad \theta = \pi \quad x^T y = -\|x\| \cdot \|y\| \quad y = -\alpha x$

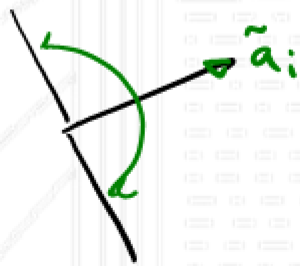


$x \perp y$ - perpendicular $\quad \theta = \pm \frac{\pi}{2} \quad x^T y = 0.$



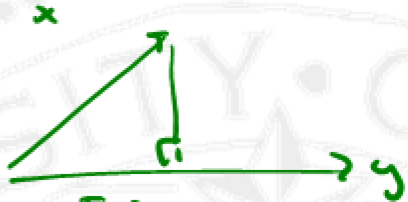
$y = Ax$ — all y_i's are > 0.

$\langle \tilde{a}_i, x \rangle$



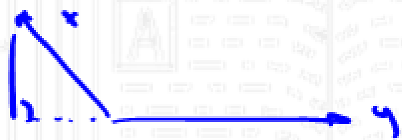
Half-space Interpretation

$x^T y > 0$ means $\angle(x,y)$ acute $\quad \measuredangle\theta \le 90^\circ$



$x^T y > 0$

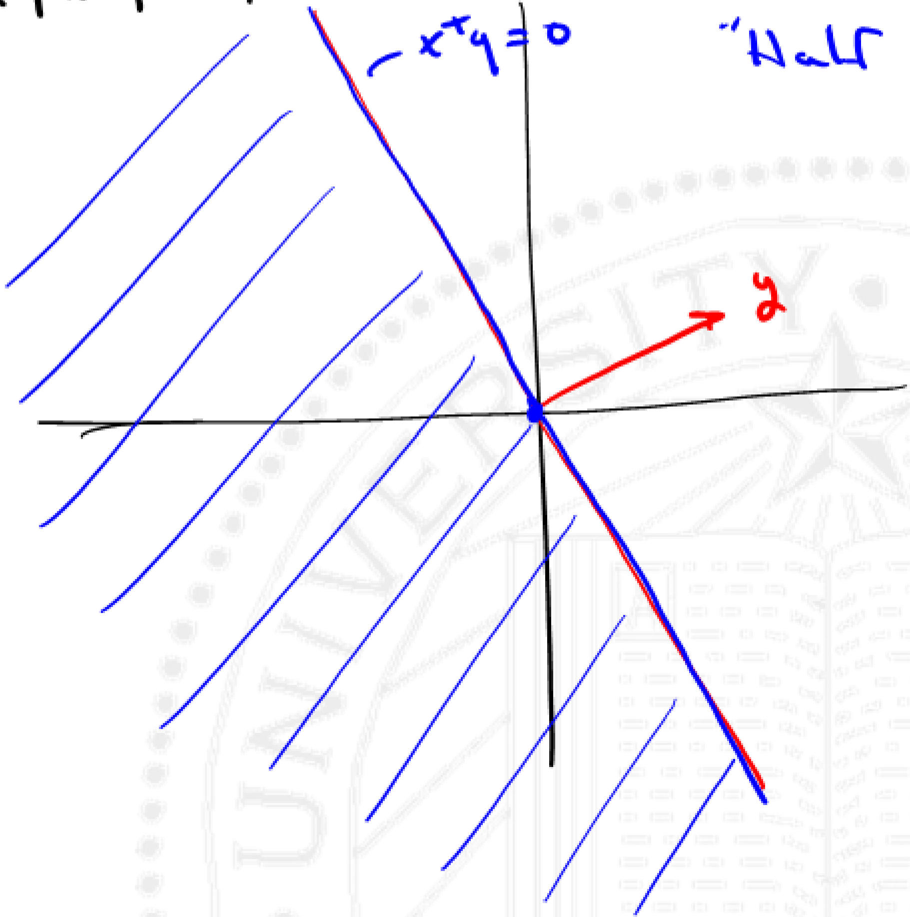
$x^T y < 0$ means $\angle(x,y)$ obtuse



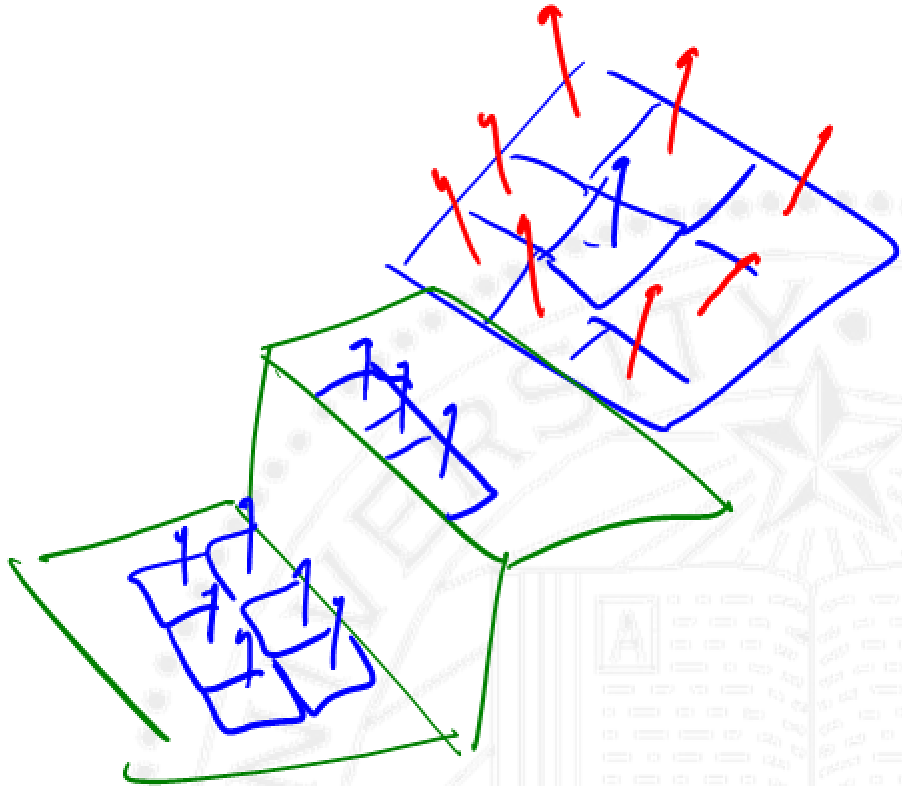
$x^T y < 0$

$\{x \mid x^T y \leq 0\}$

"Half space"



Questions?



Orthonormal Vectors and QR Factorization

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Winter 2016

Orthonormal Vectors and QR Factorization

- Orthonormal Vectors → orthogonal matrix
- Gram-Schmidt Procedure, QR Factorization
- Orthogonal Decomposition induced by a matrix

Orthonormal Set of Vectors

Set of vectors $\{u_1 \ldots u_k\} \in \mathbb{R}^n$

- Normalized is $\|u_i\| = 1$ $i = 1 \ldots k$ length of 1.

(u_1 – unit vectors, direction vectors)

- Orthogonal if $u_i \perp u_j$ for $i \neq j$

$\langle u_i, u_j \rangle = 0$ $u_i^T u_j = 0$ $\forall i \neq j$ $UU^T \neq I$

- Orthonormal if BOTH

$U = [u_1 \ldots u_k] \in \mathbb{R}^{n \times k}$ $\underbrace{U^T U}_{\text{gram matrix}} = I_k$

if $\underline{k < n}$ then $UU^T \neq I$

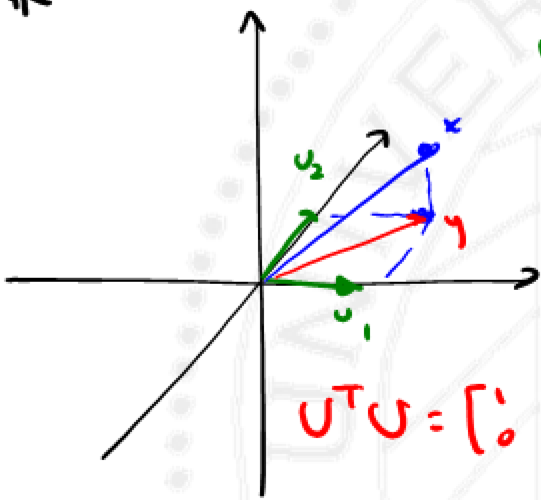
$\mathbb{R}^2$

$y = Ax$

$\mathbb{R}^3$

$$n\begin{bmatrix} u_1 \end{bmatrix} \qquad \sum^k \begin{bmatrix} u_i \end{bmatrix}\begin{bmatrix} u_i^T \end{bmatrix} = \text{Never be } I$$

$\mathbb{R}^3$



$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$U^TU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

$$UU^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\neq I_{3\times 3}$$

$U \doteq [u_1 \ldots u_k]$ called orthogonal if

$\{u_1 \ldots u_k\}$ are orthonormal

Geometric Properties

orthonormal vectors are independent

$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_k u_k = 0 \Leftrightarrow \alpha_i = \phi$.

$U = [u_1 \dots u_k]$ orthonormal basis for $\mathbb{R}^n$

$\text{span}\{u_1 \dots u_k\} = \mathcal{R}(U)$

$U \in \mathbb{R}^{n \times k}$

if $\Rightarrow w = U z$ ($z \in \mathbb{R}^k$, $w \in \mathbb{R}^n$) $\longrightarrow \|w\| = \|z\|$

MULTIPLICATION BY U preserves Length.

Mapping $w = Uz$ is ISOMETRIC preserves distance

$$\|w\|^2 = w^T w = (Uz)^T(Uz) = z^T \underbrace{U^T U}_{I} z = z^T z = \|z\|^2$$

string

→ inner product (angles) are also preserved

$$\langle Uz, U\tilde{z} \rangle = \langle z, \tilde{z} \rangle$$

unoriginal

Orthonormal Basis for R^n

$\underline{k = n}$

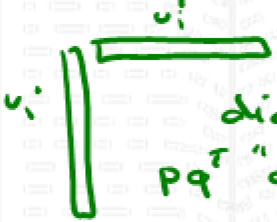
$\{u_1 \dots u_n\}$ orthonormal basis for $\mathbb{R}^n$

$U = [u_1 \dots u_n]$ orthogonal $\in \mathbb{R}^{n \times n}$

$$U^T U = I$$
$$U U^T = I \Rightarrow \underline{U^{-1} = U^T}$$

$$\sum_{i=1}^{n} u_i u_i^T = I$$

n $\quad$ $n \times n$ Rank 1 matrix



diadic expansion

pq^T "diad" outer product

Expansion in Orthonormal Basis

U orthogonal $\quad x = UU^T x \quad [u_1 \cdots u_n]\begin{bmatrix} u_1^T x \\ \vdots \\ u_n^T x \end{bmatrix}$

$$x = \sum_{i=1}^{n} (u_i^T x) u_i$$

$u_i^T x$ component of x in the direction of u_i

$a = U^T x$ resolves x into its u_i components

$x = Ua$ reconstructs x from its u_i components

$$x = Ua = \sum_{i=1}^{n} a_i u_i$$ called u_i expansion of x

$$I = UU^T = \sum_{i=1}^{n} u_i u_i^T$$

$$I = \sum_{i=1}^{n} |u_i\rangle\langle u_i|$$

$$x = \sum_{i=1}^{n} |u_i\rangle\langle u_i|x\rangle$$

Rotation Matrix in $\mathbb{R}^3$ → $\underline{SO(3)}$.

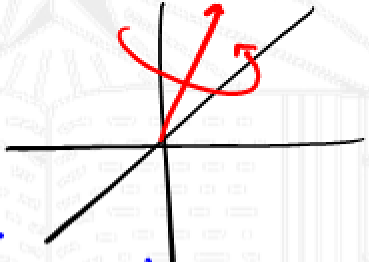
Geometric Interpretation

U orthogonal $\in \mathbb{R}^{n\times n}$ $\qquad w = Uz$.

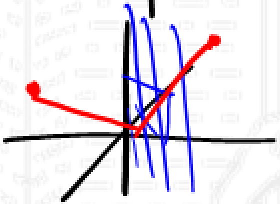
$\|Uz\| = \|z\|$, $\angle(Uz, U\tilde{z}) = \angle(z, \tilde{z})$

<u>examples</u>

Rotation about some axis



Reflection through some plane

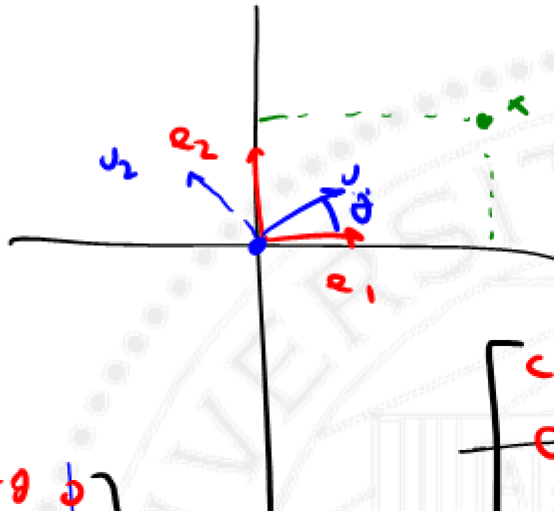


Examples: Rotation/Reflection

$y = U_\theta x$

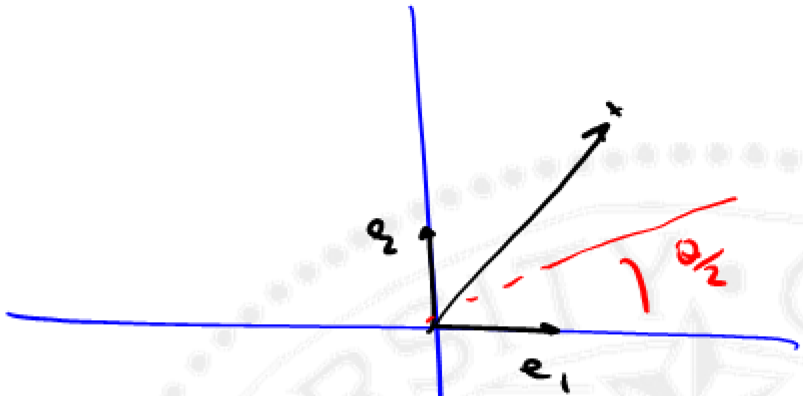
$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotation θ degrees



$$\begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

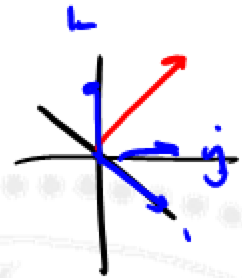


$$y = R_\theta x$$

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Quaternion

$$\begin{bmatrix} q_0 \\ \cdots \\ q_1 \\ | \\ q_3 \end{bmatrix} \quad \begin{matrix} \sin\left(\frac{\theta}{2}\right) \\ \\ \left.\right\} \hat{e}\cos\left(\frac{\theta}{2}\right) \end{matrix}$$



$$q^T q = 1.$$

$$\dot{q} = -\frac{1}{2}\,\Omega\, q$$

$$\begin{bmatrix} \omega^T \\ (\omega\times) \quad -\omega^T \end{bmatrix}$$

Gram-Schmidt Procedure (1.3)

Given an independent set of vectors $\{a_1 \ldots a_k\} \in \mathbb{R}^n$

G-S finds $q_1 \cdots q_k$ such that

$$\text{span}\{a_1 \ldots a_r\} = \text{span}\{q_1 - q_r\} \quad r \le k$$

$\{q_1 \cdots q_r\}$ orthonormal basis for $a_1 \cdots a_r$

$$A \in \mathbb{R}^{n \times k} \longrightarrow R(A) = R(Q)$$

$\sim$ 200 years old.

Gram-Schmidt Procedure (2.3)

$\tilde{q}_1 := a_1$

$q_1 := \tilde{q}_1 / \|\hat{q}_1\|$ ← normalize.

$\tilde{q}_2 := a_2 - (q_1^T a_2) \cdot q_1$

$q_2 := \tilde{q}_2 / \|q_2\|$

$q_3 := a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$