Matrix Multiplication as Mixture of Columns

$$y = Ax \qquad A \in \mathbb{R}^{m \times n}$$

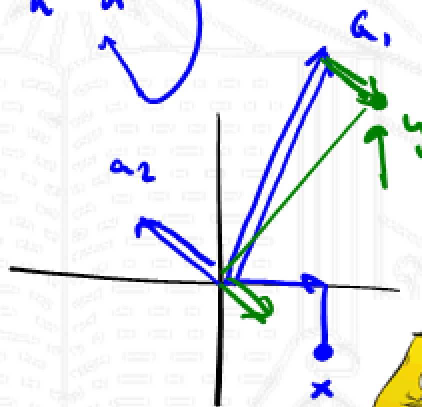
$$A = [a_1 \; a_2 \; \cdots \; a_n] \qquad a_j \in \mathbb{R}^m$$

$$y = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$$

x_j's are scalars

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$y = Ax = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$

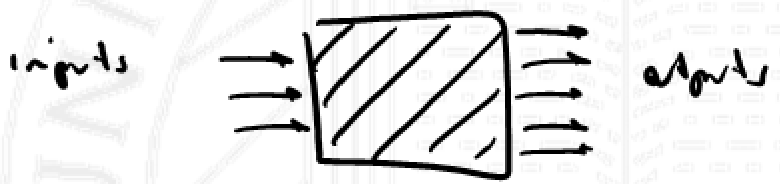


Unit Vectors

$x = e_j$ the j^{th} unit vector

$$e_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad e_n \triangleq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$Ae_j = a_j \leftarrow j^{th}$ column of A.



e_j on input extracts column of A on output

Matrix multiplication as inner product with rows

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_n^T \end{bmatrix}$$

row → gain from output to all inputs

$$y = Ax = \begin{bmatrix} \tilde{a}_1^T x \\ \tilde{a}_2^T x \\ \vdots \\ \tilde{a}_n^T x \end{bmatrix}$$

$\langle \tilde{a}_1, x \rangle$

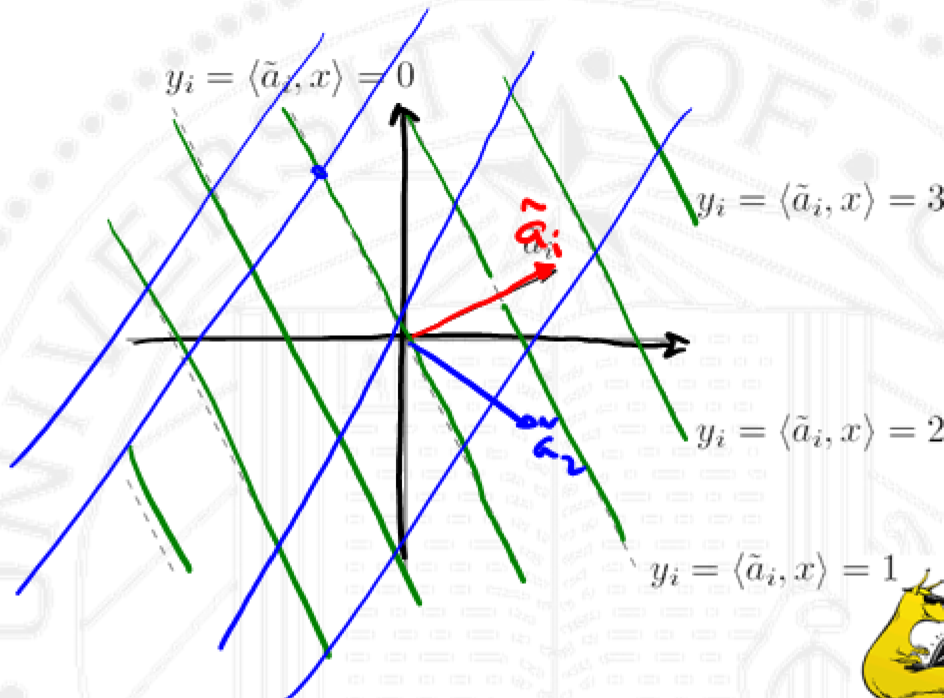
inner product

y_i → inner product of $\langle \tilde{a}_i, x \rangle$

$$y_i = \tilde{a}_i^T x = \langle \tilde{a}_i, x \rangle = \alpha \text{ in } \mathbb{R}^n$$

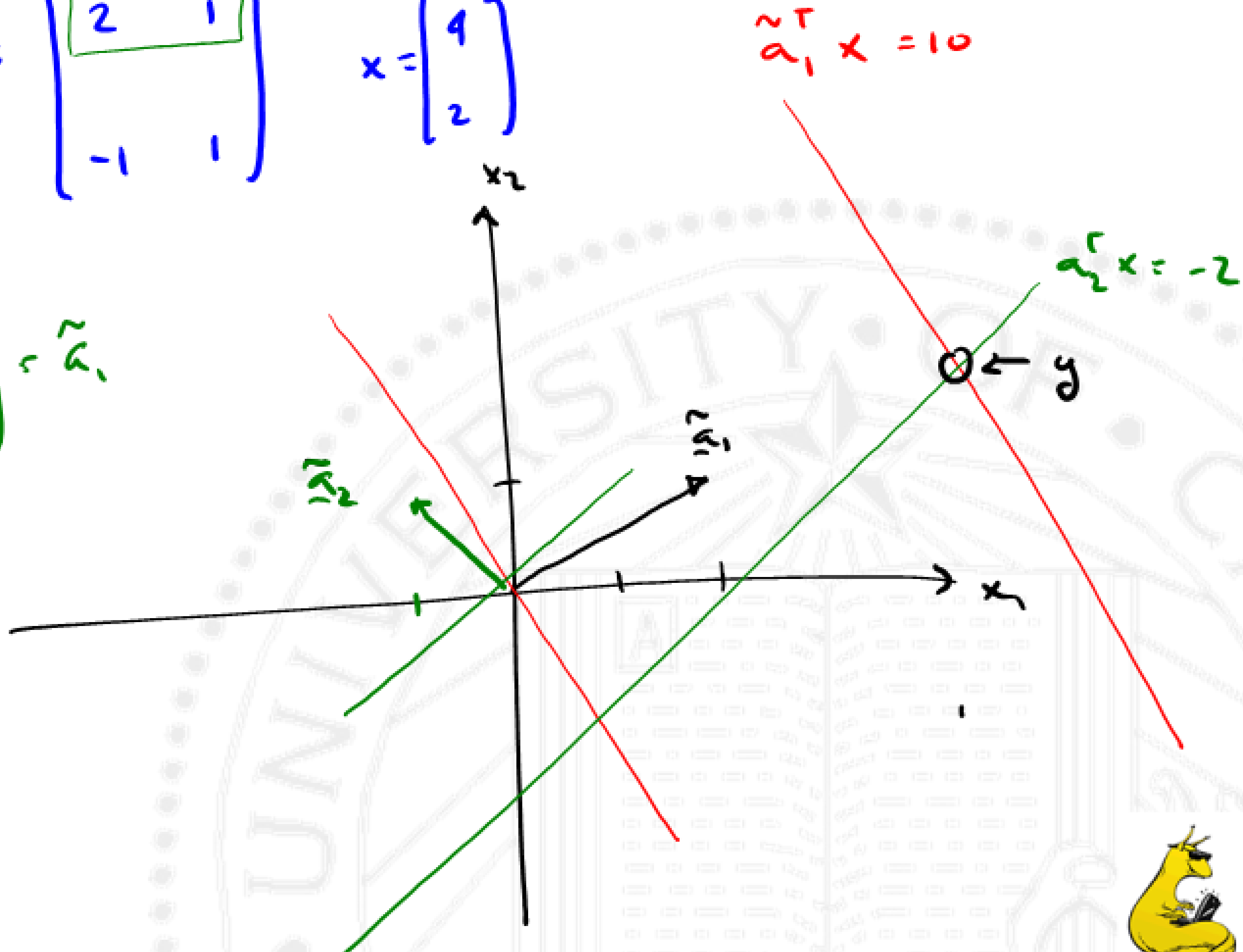
Geometric Interpretation

$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



$$A = \begin{bmatrix} \boxed{2 \quad 1} \\ -1 \quad 1 \end{bmatrix} \qquad x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

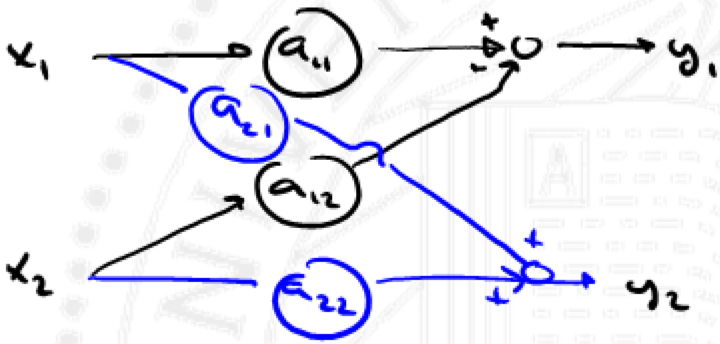
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \tilde{a}_1$$



Block Diagram Representation

$y = Ax$ signal flow graph or block diagram

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Full bipartite graph

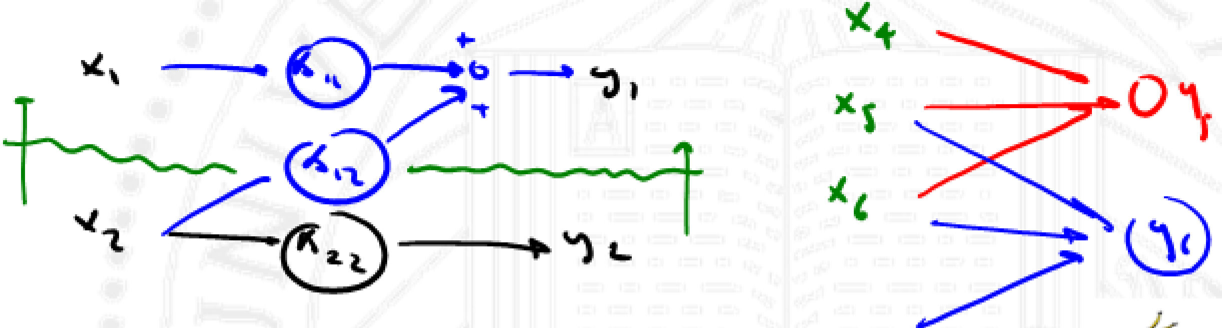
Example: Block Upper Triangular

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \phi & A_{22} \end{bmatrix}$$

$A_{11} \in \mathbb{R}^{m_1 \times n_1}$ ...

$y_1 = A_{11} x_1 + A_{12} x_2$

$y_2 = A_{22} x_2$



Matrix Multiplication as Composition

inner dimension MUST match

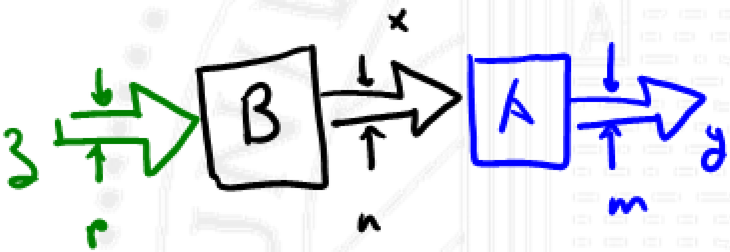
$A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times p}$ $\quad C \triangleq AB \in \mathbb{R}^{m \times p}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\boxed{\begin{array}{c} \sim O(n^3) \\ (k\, O\, n^{?}) \end{array}}$$

$y = Cz \rightarrow y = Ax \quad x = Bz.$

$Cz = A(Bz)$



Column and Row Interpretations

$$C = AB = [Ab_1 \; Ab_2 \cdots Ab_p]$$ parallel vector/matrix multiplies

$$C = AB = \begin{bmatrix} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{bmatrix}$$

Inner Product Interpretation

$$c_{ij} = \tilde{a}_i^T b_j = \langle \tilde{a}_i, b_j \rangle$$

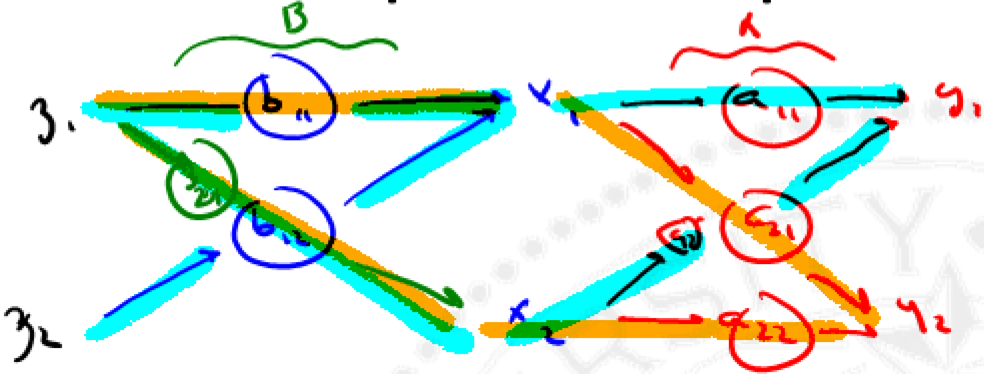
$c_{ij} = \phi$ ith row of A $\perp$ j colm of B

<u>Gram Matrix</u> $f_1 \ldots f_n$ $\quad G_{ij} = f_i^T f_j$

$$G = [f_1 \ldots f_n]^T [f_1 \ldots f_n]$$

$$AB = I \longrightarrow \tilde{a}_i^T b_j = \begin{cases} 1 & i = j \\ \phi & \text{otherwise} \end{cases}$$

Matrix Multiplication Interpretation via Paths



c_{ij} total path gain from input j to output i

Questions?

Linear System: $y = Ax$

Affine System: $y = Ax + b$

define $\tilde{y} \hat{=} (y - b)$

$\underline{\tilde{y} = Ax}$

Linear Algebra Review

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Winter 2016

Linear Algebra Review

- Vector Spaces, subspaces
- Independence, basis, dimension
- Range, nullspace, rank
- Change of coordinates
- Norm, angle, inner product

Vector Spaces

- A *vector space* or *linear space* consists of:

- a set $\mathcal{V}$ "$x+y$"
- a vector sum $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ $\quad +(x,y)$
- a scalar multiplication : $\mathbb{R} \times \mathcal{V} \to \mathcal{V}$ $\quad \times(\alpha, v)$ "αv"
- a distinguished element $0 \in \mathcal{V}$

Vector Space Properties

$x + y = y + x \quad \forall\, x, y \in \mathcal{V}$ (+ commutative)

$(x + y) + z = x + (y + z) \quad \forall\, x, y, z \in \mathcal{V}$ (+ associative)

$0 + x = x \quad \forall\, x \in \mathcal{V}$ (additive identity)

$\forall x \in \mathcal{V} \quad \exists\, (-x) \in \mathcal{V}$ such that $x + (-x) = 0$

(existence of an additive inverse)

$(\alpha\beta)x = \alpha(\beta x) \quad \forall\, \alpha, \beta \in \mathbb{R}$ (scalar multiplication associative)

$\alpha(x + y) = \alpha x + \alpha y$ (right distributive rule)

$(\alpha + \beta)x = \alpha x + \beta x$ (left distributive rule)

$1x = x \quad \forall\, x \in \mathcal{V}$

Vector Space Examples

$V_1 = \mathbb{R}^n$ w/ standard elementwise vector addition and scalar multiplication

$V_2 = \{0\}$ where $0 \in \mathbb{R}^n$ ← trivial

$V_3 = \text{span}\{v_1, v_2 \ldots v_k\}$ where

$$\text{span}\{v_1, v_2 \ldots v_k\} \triangleq \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$$

$v_1 \ldots v_k \in \mathbb{R}^n$

Vector Spaces of Functions

$V_4 = \{ x : \mathbb{R}_+ \to \mathbb{R}^n \mid x \text{ is differentiable} \}$ where vector sum is a sum of functions:

$$(x+z)(t) = x(t) + z(t)$$

scalar multiplication defined as:

$$(\alpha x)(t) = \alpha x(t)$$

a point in V_4 is a trajectory in $\mathbb{R}^n$

Subspaces

A subspace of a vector space is a subset of a vector space which is itself also a vector space.

Subspace is closed under vector addition and scalar multiplication

V_1, V_2, V_3 are all subspaces of $\mathbb{R}^n$

$$\boxed{V_5 = \{ x \in V_4 \mid \dot{x} = Ax \}}$$

points in V_5 are trajectories of $\dot{x} = Ax$

Independent Set of Vectors

Property of a set of vectors $\{v_1 \dots v_k\}$ is independent

is:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_k v_k = 0 \Rightarrow \alpha_1 \dots \alpha_k = 0$$

$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_k v_k$ are uniquely determined

$$\alpha_1 v_1 + \dots + \alpha_k v_k = \beta_1 v_1 + \dots + \beta_k v_k \rightarrow \alpha_1 = \beta_1 \dots \alpha_k = \beta_k$$

No vector v_i can be expressed as a linear combination of the other vectors.

ex: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Basis and Dimension

$\{v_1 \ldots v_k\}$ is a basis for a vector space V if:

$v_1 \ldots v_k$ span V $\quad V = \text{span}\{v_1 \ldots v_k\}$

and $v_1 \ldots v_k$ are independent

every $v \in V$ can be uniquely expanded

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_k v_k$$

$\text{Dim}(V) \triangleq$ # of v_k's that form a basis.

"Cardinality"

Nullspace of a Matrix

$A \in \mathbb{R}^{m \times n}$ nullspace "kernel" $N(A) \triangleq \{x \in \mathbb{R}^n \mid Ax = 0\}$ $\mathbb{R}^m$

$N(A)$ is a set of vectors mapped to zero by $y = Ax$

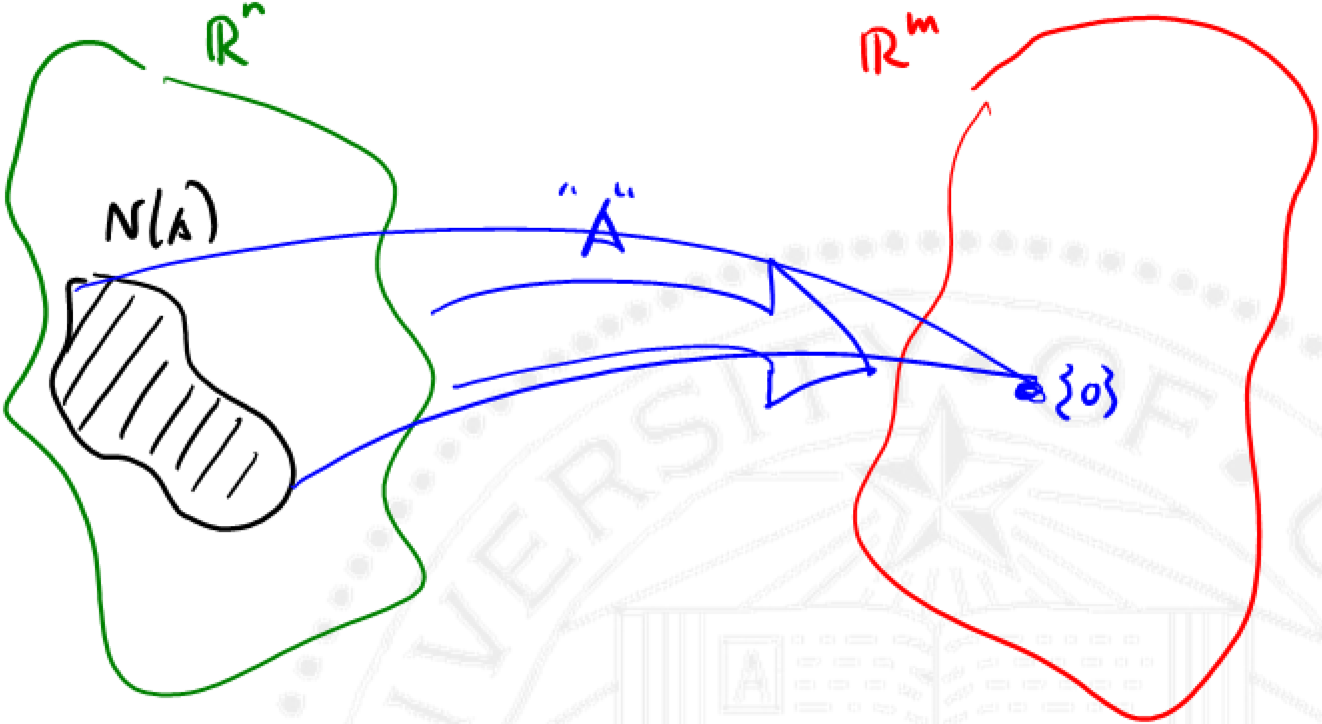
$N(A)$ is a set of vectors that are orthogonal to all rows of A

$y_i = \tilde{a}_i^T x$

$N(A)$ gives the ambiguity in x given that $y = Ax$

$y = Ax$ $z \in N(A)$ then $y = A(x + z)$

$y = Ax$ $y = A\tilde{x}$ then $\tilde{x} = x + z$ for some $z \in N(A)$



$\mathbb{R}^n$
$\mathbb{R}^m$
N(A)
"A"
{0}

Zero Nullspace

$\mathbb{R}^n$

$N(A) = \{0\}$ — "one to one"

$\underline{x}$ can always be determined from $y = Ax$

$y = Ax$ transformation loses no information

map from $x \to y$ different x's map to different y's.

$\det(A^T A) \neq 0$.

perfect decoder.

A has a LEFT inverse $B \in \mathbb{R}^{n \times m}$ such that $BA = I$

$y = Ax \quad x = By = BAx = Ix$.

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Interpretations of Nullspace

$y = Ax$ $\qquad z \in N(A)$

$y = Ax$ is a measurement → y measurable by many sensors.

x and $x+z$ are indistinguishable

$N(A)$ ambiguity in x from measurement $y = Ax$

$y = Ax$ output from input v

z is input w/ no result.

$x+z$, x have the same result

$N(A)$ freedom of choice in x.