Symmetric Matrices, Quadratic Forms, Matrix Norm, and SVD

Gabriel Hugh Elkaim

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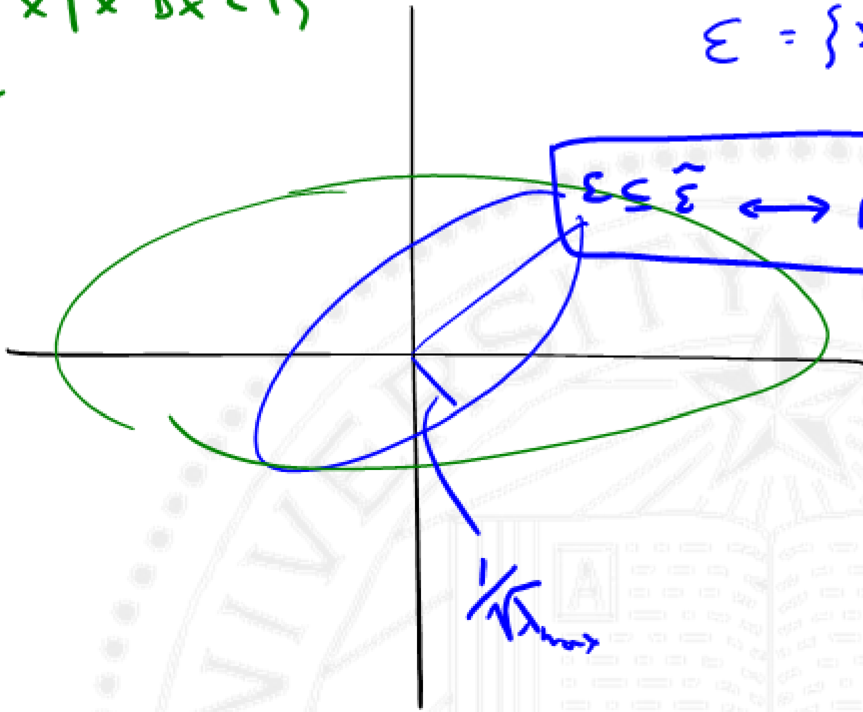
$\tilde{\mathcal{E}} = \{x \mid x^T B x < 1\}$

$\mathcal{E} = \{x \mid x^T A x < 1\}$

$\mathcal{E} \subseteq \tilde{\mathcal{E}}$

$\boxed{\mathcal{E} \subseteq \tilde{\mathcal{E}} \longleftrightarrow A \geq B}$



Gain of a Matrix in a Direction (1.3)

$A \in \mathbb{R}^{m \times n}$ not necessarily square or symmetric

$y = Ax$ $\quad x \in \mathbb{R}^n$ $\quad \dfrac{\|Ax\|}{\|x\|}$ gain the amplification factor

gain of A in direction of x.

Gain of a Matrix in a Direction (2.3)

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$ spectral norm of A

L_2 norm of A $\|A\|$

$\lambda_{max}(A^TA)$

$$\max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{(Ax)^T(Ax)}{x^Tx} = \frac{x^T(A^TA)x}{x^Tx} = \underline{\lambda_{max}(A^TA)}$$

$$\min_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \min_{x \neq 0} \frac{(Ax)^T(Ax)}{x^Tx} = \frac{x^T(A^TA)x}{x^Tx} = \lambda_{min}(A^TA)$$

Gain of a Matrix in a Direction (3.3)

$$\sqrt{\lambda_{max}(A^TA)} \geq \|A\| \geq \sqrt{\lambda_{min}(A^TA)}$$

$A^TA \in \mathbb{R}^{n\times n}$ is symmetric $A^TA \geq 0$

so that $\lambda_{max}, \lambda_{min} \geq 0$

max gain $x = q_1$ eig(A^TA)

min gain $x = q_n$ eig(A^TA)

Matrix Norm (1.4)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$e_1 \to x^T A x \to \sim 6$

$e_2 \to x^T A x \to \sim 7$

$$A^T A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix} \to \begin{bmatrix} 0.62 & .785 \\ .785 & -0.62 \end{bmatrix} \begin{bmatrix} 90.7 & 0 \\ 0 & 0.265 \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix}^{*}$$

$$\|A\| = \sqrt{\lambda_{max}(A^T A)} = \underline{9.53}$$

$x \to \begin{bmatrix} 0.62 \\ .785 \end{bmatrix} \to \|x\| = 1 \quad \underline{\|Ax\| \cong 9.53}$

$x = \begin{bmatrix} .785 \\ -.62 \end{bmatrix} \quad \|x\| = 1 \quad \|Ax\| \cong \underline{0.5}$

Matrix Norm (2.4)

min gain $\sqrt{\lambda_{min}(A^TA)} = 0.514$

$x = \begin{bmatrix} .785 \\ -.62 \end{bmatrix}$ — $\|x\| = 1$ $\|Ax\| = 0.514$

$0.514 \leq \|A\| \leq 9.53$

Matrix Norm (3.4)

Consistent w/ vector norm

$a \in \mathbb{R}^{n \times 1} \rightarrow \sqrt{\lambda_{max}(a^T a)} = \sqrt{a^T a} = \|a\|$

for any x $\quad \|Ax\| \leq \|A\| \|x\|$ — along direction of eig $(A^T A)$.

$\|aA\| = |a| \|A\|$

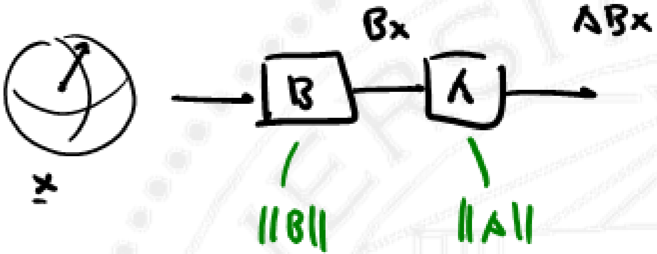
$\|A + B\| \leq \|A\| + \|B\|$

$\|A\| = \phi \rightarrow A = \phi$.

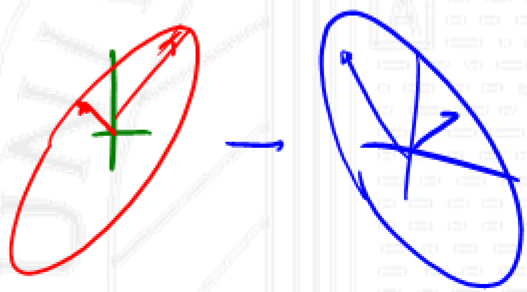
$\boxed{\|AB\| \leq \|A\| \cdot \|B\|}$.

Matrix Norm (4.4)

geometric intuition



$$\frac{x^T (AB)^T (AB) x}{x^T x}$$



Properties of Matrix Norm (3.3)

$A \in \mathbb{R}^{n \times n} \longrightarrow A = T \Lambda T^{-1}$ eigenvalue / eigenvector

$A \in \mathbb{R}^{n \times n} \quad A = A^T \longrightarrow A = Q \Lambda Q^T$ symmetric eigenvector

$A \in \mathbb{R}^{n \times m} \longrightarrow$ SVD: $A = U \Sigma V^T$

Singular Value Decomposition (1.3)

SVD of A: $A = U \Sigma V^T$ $\quad A \in \mathbb{R}^{n \times m}$

$A \in \mathbb{R}^{m \times n}$ $\quad \text{rank}(A) = r$

$U \in \mathbb{R}^{m \times r}$ $\quad U^T U = I$

$V \in \mathbb{R}^{n \times r}$ $\quad V^T V = I$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \quad \sigma_1 \geq \sigma_2 \cdots \geq \sigma_n$$

singular values of A.

Singular Value Decomposition (2.3)

$$\underset{m}{\left[\overset{n}{A}\right]} = \underset{m}{\left[\overset{r}{U}\right]}\left[\overset{r\times r}{\Sigma}\right]\left[\overset{n}{V^T}\right]_r$$

$U = [u_1 \cdots u_r]$

$V = [v_1 \cdots v_r]$

$$A = U\Sigma V^T = \sum_{i=1}^{r} \underbrace{\sigma_i u_i v_i^T}_{\substack{\text{rank 1} \\ \text{dyadic form}}} \sim \lambda_i v_i w_i^T$$

$$\|A\| = \sigma_1 \text{ or } \sigma_{max}$$

Singular Value Decomposition (3.3)

$\sigma_i \triangleq$ singular value of A

$v_i \triangleq$ right (input) singular vector of A

$u_i \triangleq$ left (output) singular vector of A

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma \underbrace{U^T U}\Sigma V^T = V\Sigma^2 V^T$$

$v_i \triangleq$ eigenvectors of A^TA corresponding to non-zero eigenvalues

$\sigma_i \triangleq \sqrt{\lambda_i(A^TA)}$ $\quad \lambda_i(A^TA) = 0 \;\; \forall\, i > r$

Interpretations (1.3)

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma \underline{V^T V}\Sigma U^T = U\Sigma^2 U^T$$

u_i are eigenvectors of (AA^T) associated with nonzero eigenvalues

$$\text{eig}(AB) = \text{eig}(BA)$$

pq^T — q^Tp scalar.

$\sigma_i = \sqrt{\lambda_i(AA^T)}$ and $\lambda_i(AA^T) = 0$ $\underline{i > r}$.

Interpretations (2.3)

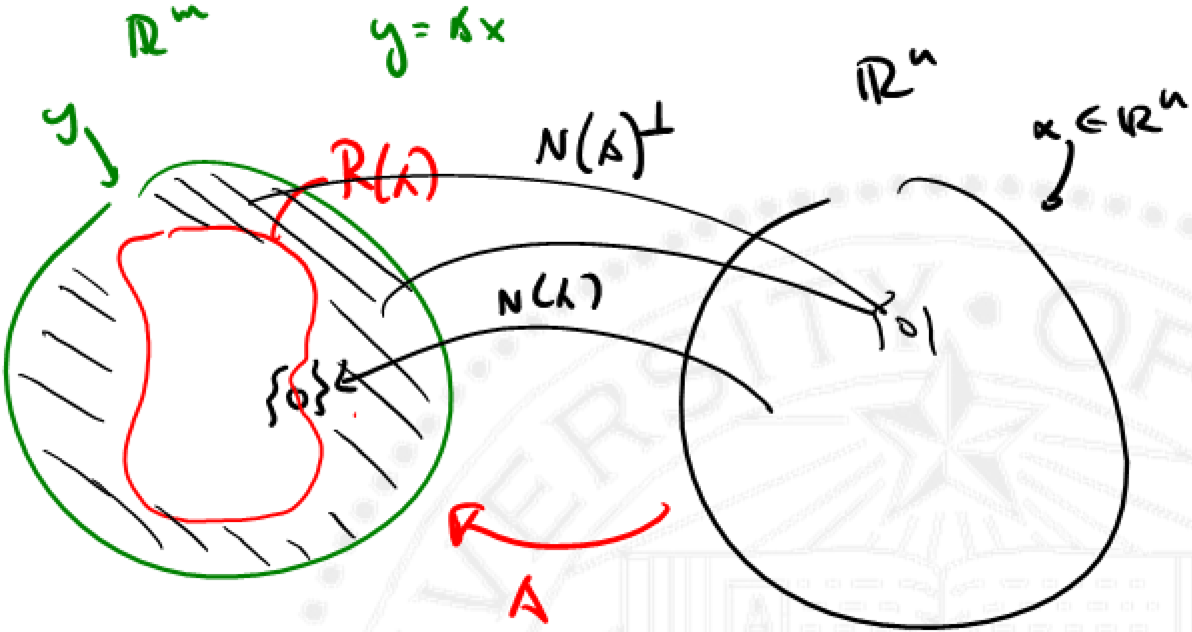
$u_i \to$ eigenvectors of (AA^T) LEFT/OUTPUT directions

$v_i \to$ eigenvectors of (A^TA) RIGHT/INPUT direction.

$\sigma_i \to$ eigenvalues of $\sqrt{\lambda_i(A^TA)} = \sqrt{\lambda_i(AA^T)}$

$[u_1 \cdots u_r]$ orthonormal basis for $R(A)$

$[v_1 \cdots v_r]$ orthonormal basis for $N(A)^\perp$



$\mathbb{R}^m$
$y = Ax$
y
$R(A)$
$N(A)^{\perp}$
$\mathbb{R}^n$
$x \in \mathbb{R}^n$
$N(A)$
$\{0\}$
0
A

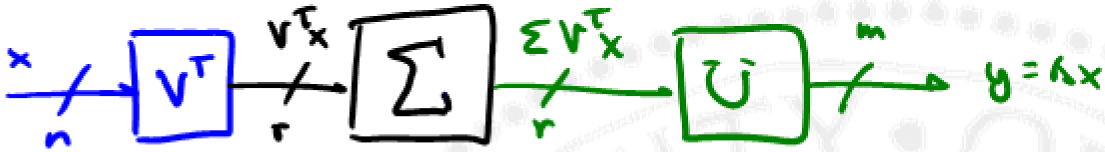
Interpretations (3.3)

$R(U) = R(A)$

Any linear mapping

$N(A)^{\perp} = R(V)$

$$A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad y = Ax$$



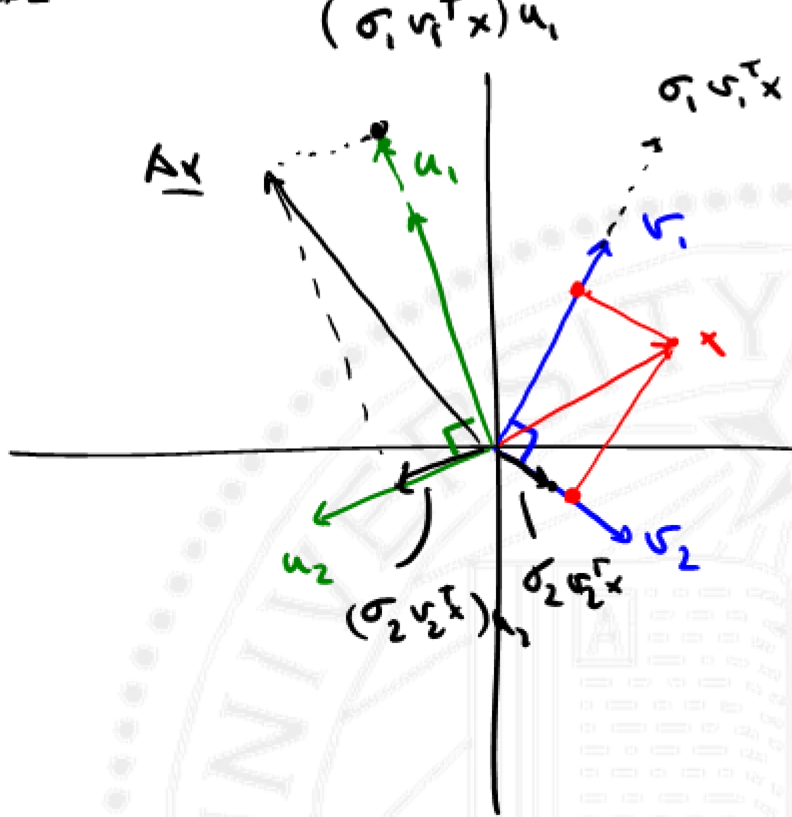
resolve x into v_i input directions

dilate along coordinate axes

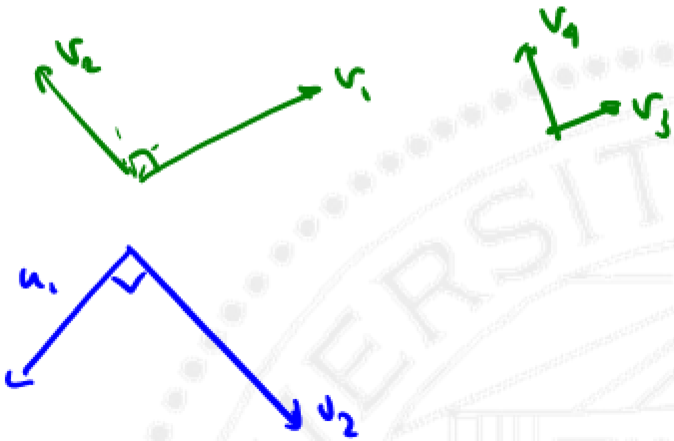
reshape results back along output directions

- Compute coeff. of x along input direction $v_1 \ldots v_r$
- Scale these coefficients by σ_i
- reconstruct y along the output directions.

$A \in \mathbb{R}^{2\times 2}$



$A \in \mathbb{R}^{4\times 4}$ $\qquad \Sigma = \mathrm{diag}(10, 7, 0.1, 0.05)$



$y = Ax$

$$10 \geq \frac{\|Ax\|}{\|x\|} \geq 0.05$$

A effectively R^2

$\dot{x} = Ax$

$y = Cx$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\mathrm{rank}(\mathcal{O})$

$\boxed{C_1}$ ~ 300

$\boxed{C_2}$ ~ 12

$\boxed{\dfrac{\sigma_{max}}{\sigma_{min}}}$ σ_1

σ_2

condition # [1 → ∞]

rcond $\dfrac{\sigma_{min}}{\sigma_{max}}$ (1 → 0)

Singular Value Decomposition Applications

Gabriel Hugh Elkaim

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Singular Value Decomposition Applications

- General pseudo-inverse
- Full SVD
- Image of Unit Ball under linear transformation
- SVD in estimation/inversion
- Sensitivity of linear equations to data error
- Low rank approximation via the SVD

General Pseudo-Inverse (1.3)

eigenvalues: $A = T\Lambda T^{-1}$ $A \in \mathbb{R}^{n \times n}$

symmetric: $A = Q\Lambda Q^T$ $A = A^T \in \mathbb{R}^{n \times n}$

svd: $A = U\Sigma V^T$ $A \in \mathbb{R}^{n \times m}$, rank r

$U^T U = I$ $[\sigma_i]$ $V^T V = I$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$

A^+ defined for skinny, fat, full rank matrices

General Pseudo-Inverse (2.3)

if A has an SVD : $A = U \Sigma V^T$

Σ ← diagonal, r×r, positive

$$A^{+} = V \Sigma^{-1} U^T \leftarrow \text{pinv}$$

skinny full rank $A \in \mathbb{R}^{n \times m}$ $\quad n \begin{bmatrix} \\ \end{bmatrix}^{m}$ $\quad n > m$

$$A^{+} = (A^T A)^{-1} A^T = ((U \Sigma V^T)^T (U \Sigma V^T))^{-1} (U \Sigma V^T)^T$$

$$\begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_m \end{bmatrix}$$

$$A^T A = (U \Sigma V^T)^T U \Sigma V^T = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$$

$$(A^T A)^{-1} = V \Sigma^{-2} V^T$$

$$(A^T A)^{-1} A^T = V \Sigma^{-2} V^T \cdot V \Sigma U^T = V \Sigma^{-2} \cdot \Sigma U^T = V \Sigma^{-1} U^T$$

General Pseudo-Inverse (3.3)

$$A^{+} = A^{T}(AA^{T})^{-1} = (U\Sigma V^{T})^{T}\left[(U\Sigma V^{T})(U\Sigma V^{T})^{T}\right]^{-1}$$

$$\rightarrow AA^{T} = U\Sigma V^{T}\cdot V\Sigma U^{T} = U\Sigma^{2}U^{T}$$

$$(AA^{T})^{-1} = U\Sigma^{-2}U^{T}$$

$$A^{+} = A^{T}(AA^{T})^{-1} = V\Sigma\underbrace{U^{T}\cdot U}\Sigma^{-2}U^{T}$$

$$= \underline{V\Sigma^{-1}U^{T}}$$

Skinny, full rank : $\begin{bmatrix} A \end{bmatrix} \longrightarrow x_{ls} = A^{+}y = V\Sigma^{-1}U^{T}y$

Fat, full rank $\begin{bmatrix} \quad A \quad \end{bmatrix} \longrightarrow x_{ln} = A^{+}y = V\Sigma^{-1}U^{T}y$.

In general

$$X_{ls} = \{ z \mid \|Az - y\| = \min_{w} \|Aw - y\| \}$$

$x_{pinv} = A^{+}y \in X_{ls}$ has min. norm on X_{ls}

Pseudo-Inverse via Regularization (1.2)

$\mu > 0$ let x_μ be a unique minimizer of

$$\|Ax_\mu - y\|^2 + \mu\|x\|^2$$

$$x_\mu = (A^TA + \mu I)^{-1}A^T y \qquad (A^TA + \mu I) > 0$$

$$x_\mu = (V\Sigma^2V^T + \mu VV^T)^{-1} V\Sigma U^T$$

$$= (V[\Sigma^2 + \mu I]V^T)^{-1} \cdot$$

$$\begin{bmatrix} \sigma_1^2+\mu & & \\ & \ddots & \\ & & \sigma_n^2+\mu \end{bmatrix}$$

$$\lim_{\mu \to 0} \longrightarrow V\Sigma^{-1}U^T = A^+$$

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Final

- Thursday — Friday

8

- Friday — Saturday

- Tuesday/Wed (3)

24 hours open everything except published solution or people.