$$\dot{x} = Ax + Bu$$

input

$$y = Cx + Du$$

Linear Dynamical Systems with Inputs and Outputs

Gabriel Hugh Elkaim

Winter 2016

Z Transform (2.3)

$Z\{v_k = w_{k+1}\}$

$$V(z) = zW(z) - z w_0 \longleftrightarrow \mathcal{L}\{\dot{x}\} = sX - x_0$$

Discrete-Time Transfer Function (1.3)

$$x_{k+1} = A x_k + B u_k$$

$$y_k = C x_k + D u_k$$

$$z X(z) - z x_0 = A X(z) + B U(z)$$

$$Y(z) = C X(z) + D U(z)$$

$$X(z) = [zI - A]^{-1} z x_0 + [zI - A]^{-1} B U(z)$$

$$Y(z) = \underline{H(z)} U(z) + C \underbrace{[zI - A]^{-1} z x_0}_{\text{I.C.}}$$

Discrete-Time Transfer Function (2.3)

$$H(z) \triangleq C[zI-A]^{-1}B+D$$

$$[zI-A]^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \dots$$

power series expansion of resolvent

Symmetric Matrices, Quadratic Forms, Matrix Norm, and SVD

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Symmetric Matrices, Quadratic Forms, Matrix Norm, and the SVD

- Eigenvectors of symmetric matrices
- Quadratic Forms
- Inequalities for quadratic forms
- Positive semi-definite matrices
- Norm of a matrix
- Singular Value Decomposition (SVD)

Eigenvalues of Symmetric Matrices (1.2)

$A \in \mathbb{R}^{n \times n}$ $\quad A = A^T$ symmetric

fact: eigenvalues of A are all real

$$Av = \lambda v \quad v \neq 0 \quad v \in \mathbb{C}^n$$

$$\bar{v}^T A v = \bar{v}^T(\lambda v) = \lambda \bar{v}^T v = \lambda \sum_{i=1}^{n} |v_i|^2$$

$$\bar{v}^T A v = (\overline{A v})^T v = (\bar{\lambda}\bar{v})^T v = \bar{\lambda}\bar{v}^T v = \bar{\lambda} \sum_{i=1}^{n} |v_i|^2$$

$\lambda = \bar{\lambda}$ i.e.: $\lambda \in \mathbb{R}$ $\quad v \in \mathbb{R}^n$

Eigenvalues of Symmetric Matrices (2.2)

"self-adjoint"

$\dot{x} = Ax$ $\quad$ $A = A^T$ $\quad$ no oscillations

choose an orthonormal set of eigenvectors of A

$q_1 \cdots q_n$ such that $Aq_i = \lambda q_i$ and $q_i^T q_j = \delta_{ij}$

$[q_1 \cdots q_n] \triangleq Q$ $\quad$ orthogonal matrix $\quad$ $Q^T = Q^{-1}$

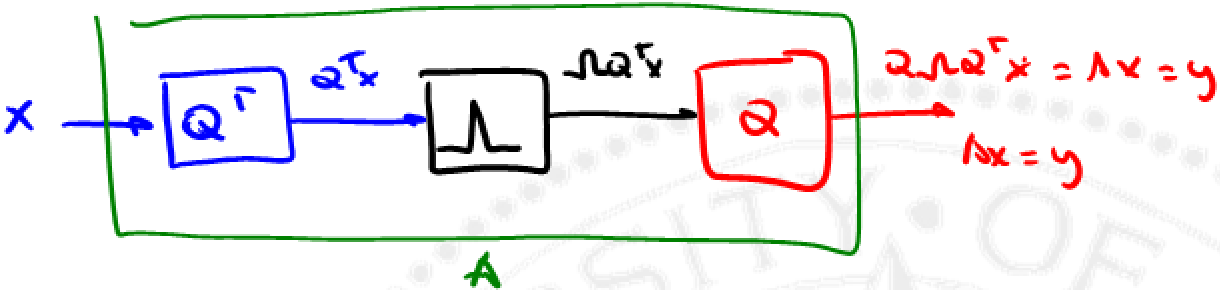
$$Q^{-1} A Q = Q^T A Q = \Lambda$$

left & right eigenvectors are the same.

$$A = Q \Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

rank 1 dyadic expansion

Eigenvectors of Symmetric Matrices (1.2)



- Resolve into q_i coordinates
- Scale those coordinates by λ_i "dilation" "dilitation"
- Reconstitute with basis q_i

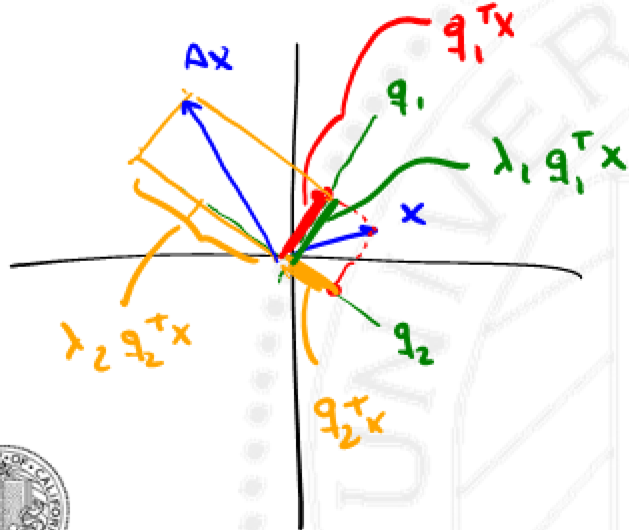
Eigenvectors of Symmetric Matrices (2.2)

$$A = \sum_{i=1}^{n} \lambda_i \underbrace{q_i q_i^T}_{n \times n \text{ rank 1 diad}}$$

orthogonal projection spanned by the line q_i

Interpretation (1.3)

$$A = \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} = \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)}_{Q} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}}_{\Lambda} \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^T}_{Q^T}$$



Interpretation (2.3)

λ_i are distinct

suppose $v_1 \cdots v_n$ is a set of linearly independent

<u>distinct</u> eigenvectors of A

$\downarrow$

$A v_i = \lambda_i v_i \qquad \|v_i\| = 1$

$$v_i^T (A v_j) = \underline{v_i^T \lambda_j v_j} = (A v_i)^T v_j = \underline{\lambda_i v_i^T v_j}$$

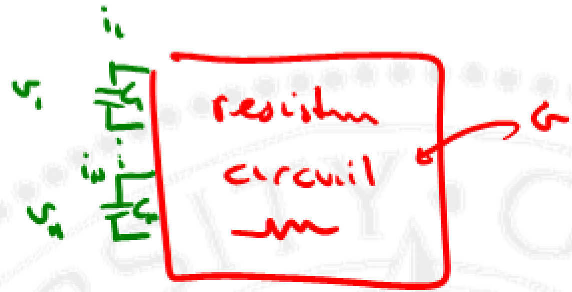
$$(\lambda_i - \lambda_j) v_i^T v_j = 0 \qquad v_i^T v_j = \phi$$

$$\underline{v_i \perp v_j}$$

Example (1.5)

$c_k \dot{v}_k = -i_k$

$i = Gv$



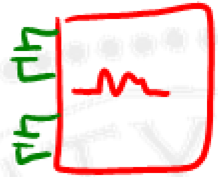
$G \triangleq$ conductance matrix

$G = G^T \in \mathbb{R}^{n \times n}$

Example (2.5)

$$\dot{v} = -\overbrace{C^{-1}G}^{\text{not symmetric}}\, v$$

$$\boxed{x_i \doteq \sqrt{c_i}\, v_i}$$



$$x = C^{1/2} v \rightarrow v = C^{-1/2} x$$

$$C = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

$$\dot{x} = C^{1/2} \dot{v}$$

$$\dot{x} = C^{1/2}\dot{v} = -\underbrace{C^{-1/2} G C^{-1/2}}_{\text{is symmetric}} x$$

$$C^{1/2} = \begin{bmatrix} \sqrt{c_1} & & \\ & \ddots & \\ & & \sqrt{c_n} \end{bmatrix}$$

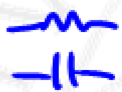
$\approx \frac{1}{2}\|x\|^2$ energy.

Example (3.5)

$C^{-1/2} G C^{-1/2}$ — symmetric → eigenvalues are real

so are eigenvalues of $\underline{-C^{-1}G}$

choose eigenvectors to be orthogonal



cannot build an oscillator

Example (4.5)

$s_i = C^{-1/2} q_i$ satisfy:

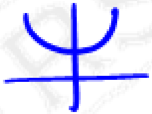
$$-C^{-1} G \underline{s}_i = \lambda \underline{s}_i \longrightarrow s_i^T C s_j = \delta_{ij}$$

Quadratic Forms (1.3)

$f : \mathbb{R}^n \to \mathbb{R}$ has the form

$$f(x) = x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

quadratic form



can assume that $A = A^T$

$$x^T A x = x^T \Big(\underbrace{\frac{A+A^T}{2}}_{\text{symmetric part of } A}\Big) x$$

Quadratic Forms (2.3)

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad x^T A x = [x_1 \; x_2]\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \; x_2]\begin{bmatrix} x_1 + 3x_2 \\ 2x_2 \end{bmatrix} = \underline{x_1^2 + 3x_1x_2 + 2x_2^2}$$

$$\frac{A + A^T}{2} = \frac{\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}}{2} = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}$$

$$x^T\left(\frac{A + A^T}{2}\right)x = [x_1 \; x_2]\begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \; x_2]\begin{bmatrix} x_1 + 3/2\, x_2 \\ 3/2\, x_1 + 2x_2 \end{bmatrix}$$

$$= x_1^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_2x_1 + 2x_2^2$$

$$= x_1^2 + 3x_1x_2 + 2x_2^2.$$

Skew symmetric $\quad \underline{S^T = -S}$

$$\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad [\omega \times]$$

cross product matrix

$$x^T S x = x^T S^T x \longrightarrow = \phi .$$

$$= - x^T S^T x$$

Quadratic Forms (3.3)

$$\|Bx\|^2 = (Bx)^T Bx = x^T \underline{B^T B}\, x$$

↑ already symmetric

$$\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \leftarrow x^T \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} x$$

measure of how wiggly x is

$$\|Fx\|^2 - \|Gx\|^2 \longrightarrow x^T (F^T F - G^T G) x$$

$\{x \mid x^T A x = a\} \longrightarrow$ quadratic surface in $\mathbb{R}^n$

$\{x \mid x^T A x \le a\} \longrightarrow$ quadratic region

$\{x \mid x^T x = 1\} \longrightarrow$ unit hypersphere in $\mathbb{R}^n$

$\le 1 \longrightarrow$ unit ball

$f(x) = x^T A x$

$x \longrightarrow \boxed{} \longrightarrow f(x)$

$y = Ax$

$e_i \longrightarrow \boxed{}$

$\begin{array}{c} \begin{bmatrix} \end{bmatrix} \\ i \end{array} \quad a_i$

$f(e_i) = e_i^T A e_i = A_{ii}$

$f(e_i + e_j) = (e_i + e_j)^T A (e_i + e_j) = A_{ii} + 2A_{ij} + A_{jj}$

Inequalities for Quadratic Forms (1.3)

$A = A^T \in \mathbb{R}^{n \times n}$ $\quad A = Q \Lambda Q^T$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

biggest $\quad$ smallest

$x^T A x = x^T Q \Lambda Q^T x = (Q^T x)^T \Lambda (Q^T x)$

$y = Q^T x$ $\quad = y^T \Lambda y = \sum \lambda_i y_i^2$

$= \sum_{i=1}^{n} \lambda_i (q_i^T x)^2 \quad \|Q^T x\|^2 = \|x\|^2$

$\lambda_1 \|x\|^2 \geq x^T A x \geq \lambda_n \|x\|^2$

$$\boxed{\lambda_{max} \|x\|^2 \geq x^T A x \geq \lambda_{min} \|x\|^2}$$

Inequalities for Quadratic Forms (2.3)

$$q_1^T A q_1 = \lambda_{max} \|q_1\|^2$$

$$q_n^T A q_n = \lambda_{min} \|q_n\|^2$$

Gain in quadratic form: $\dfrac{x^T A x}{\|x\|^2} \triangleq$ Raleigh quotient

$$\lambda_{max} \geq \frac{x^T A x}{\|x\|^2} \geq \lambda_{min}$$

Inequalities for Quadratic Forms (3.3)

$A = A^T \in \mathbb{R}^{n \times n}$

A is positive semi definite if $x^T A x \geq 0 \ \forall x$.

$\boxed{A \geq 0}$ min eig $\left(\frac{A^T + A}{2}\right) \geq 0$.

$A \geq 0$ if and only if $\lambda_{min}(A) \geq 0$

Positive semi-definite and Positive definite matrices (1.3)

$A \leq 0$ negative semi-definite $\underline{-A \geq 0}$

$A < 0$ negative definite $\underline{-A > 0}$

A $\underline{\text{indefinite}}$

Positive semi-definite and Positive definite matrices (2.3)

if $B = B^T \in \mathbb{R}^{n \times n}$ $\quad A \geq B$ if $\underline{(A-B) \geq 0}$

$\underline{A < B} \longrightarrow (A-B) < 0$ or $(B-A) > 0$

$A > B \;-\; x^T A x > x^T B x \quad \forall x \neq 0.$

Positive semi-definite and Positive definite matrices (3.3)

$A \geq B \quad C \geq D \longrightarrow A + C \geq B + D$

$B \leq 0 \longrightarrow A + B \leq A$

$A \geq 0,\ \alpha \geq 0 \longrightarrow \alpha A \geq 0$

$A \geq 0 \longrightarrow A^2 \geq 0$

$A > 0 \longrightarrow A^{-1} > 0$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \not\geq \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \not\leq \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$A \not\geq B \longrightarrow B \geq A$ <u>Bad</u>

incomparable

Matrix Inequalities (1.3)

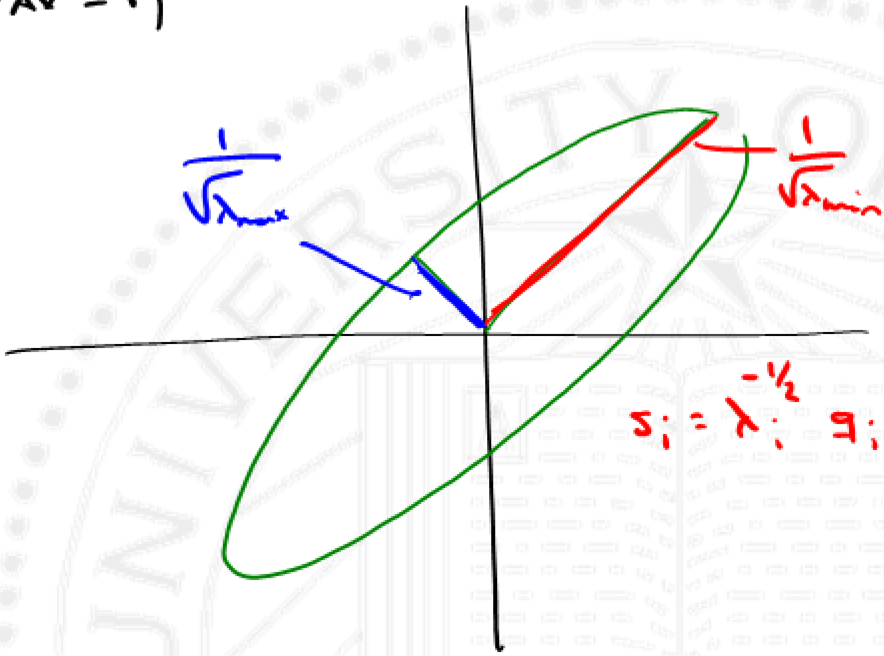
$x^T A x \le 1 \longrightarrow x^T A^{-1} x \le 1$

Ellipsoids (1.3)

$A = A^T > 0$

$\mathcal{E} = \{ x \mid x^T A x \leq 1 \}$

Ellipsoid centered @ $\{0\}$



Ellipsoids (2.3)

eigenvalues → length of the semi-major axes

eigenvectors → direction of semi-major axes

$s_i = \lambda_i^{-1/2} q_i$

q_1 $x^T A x$ is large, ellipsoid is THIN in that direction

q_n $x^T A x$ is small ellipsoid is FAT in that direction

Ellipsoids (3.3)

$\sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \triangleq$ maximum eccentricity of A

condition number.

$\tilde{\mathcal{E}} = \{x \mid x^T B x \leq 1\}$ $\quad B > 0$ $\qquad \mathcal{E} \subseteq \tilde{\mathcal{E}} \longleftrightarrow \underline{A \geq B}$

Gain of a Matrix in a Direction (1.3)

$A \in \mathbb{R}^{m \times n}$ not necessarily square or symmetric

$y = Ax \quad x \in \mathbb{R}^n \quad \dfrac{\|Ax\|}{\|x\|}$ gain the amplification factor

gain of A in direction of x.

Gain of a Matrix in a Direction (2.3)

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

spectral norm of A

L_2 norm of A

$\lambda_{max}(A^T A)$

Singular Value Decomposition (1.3)

SVD of A:

$$A = U \Sigma V^T$$

$A \in \mathbb{R}^{m \times n}$ $\quad$ $\text{rank}(A) = r$

$U \in \mathbb{R}^{m \times r}$ $\quad$ $U^T U = I$

$V \in \mathbb{R}^{n \times r}$ $\quad$ $V^T V = I$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

$\sigma_1 \ge \sigma_2 \cdots \ge \sigma_r$

singular values of A.