$$\dot{x} = Ax + Bu$$

input

$$y = Cx + Du$$

Linear Dynamical Systems with Inputs and Outputs

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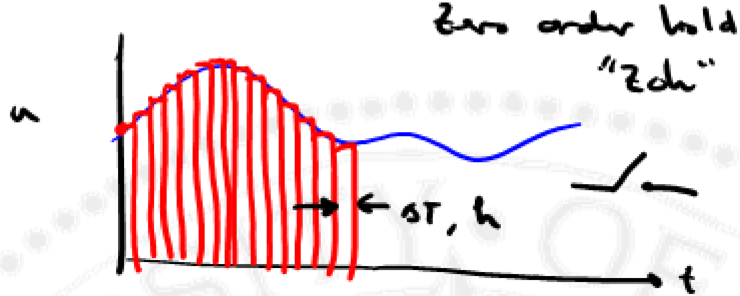
Discretization with Piecewise Constant Inputs (1.3)

$\dot{x} = Ax + Bu$

$y = Cx + Du$



$u_d : \mathbb{Z}_+ \rightarrow \mathbb{R}^m$ is an input sequence

$u(t) = u_d(k)$ for $kh \leq t \leq (k+1)h$

$x_d(k) = x(kh)$ $\quad y_d(k) = y(kh)$ $\quad k = 0, 1, \ldots$

$h > 0$ sample interval (ΔT)

u is piecewise constant (zero order hold)

Discretization with Piecewise Constant Inputs (2.3)

constant

$$x_d(k+1) = x((k+1)h)$$

$$= e^{Ah} x(kh) + \int_0^h e^{A\tau} B \overbrace{(u((k+1)h-\tau)}\, d\tau$$

$$= \underbrace{e^{Ah}}_{A_d} x_d(k) + \underbrace{\left(\int_0^h e^{A\tau} B\, d\tau\right)}_{B_d} u_d(k)$$

$$\dot{x} = Ax + Bu$$

NOT AN APPROX

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k)$$

$$y_d(k) = C x_d(k) + D u_d(k)$$

Discretization with Piecewise Constant Inputs (3.3)

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Gamma \underline{u}_k$$

$$\underline{y}_k = C \underline{x}_k + D \underline{u}_k$$

$$\Phi = e^{A \Delta T}$$

$$\Gamma = \left[\int_0^{\Delta T} e^{A\tau} \, d\tau \right] B$$

if A is invertible:

$$\underline{\int_0^{\Delta T} e^{A\tau} \, d\tau \, B = A^{-1} \left(e^{A \Delta T} - I \right) B}$$

<u>STABILITY IS PRESERVED</u>

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = H x_k + D u_k$$

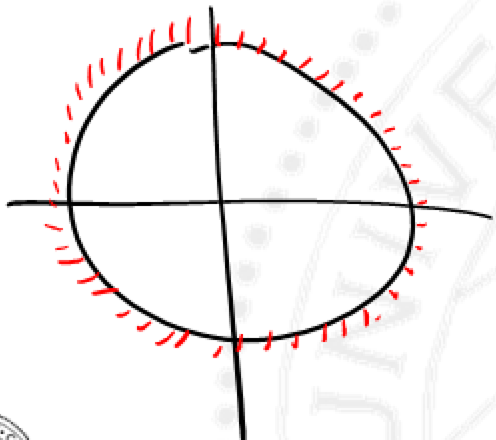
$\Phi \Leftrightarrow A_d$ $\Gamma \Leftrightarrow B_d$

$H \Leftrightarrow C_d$

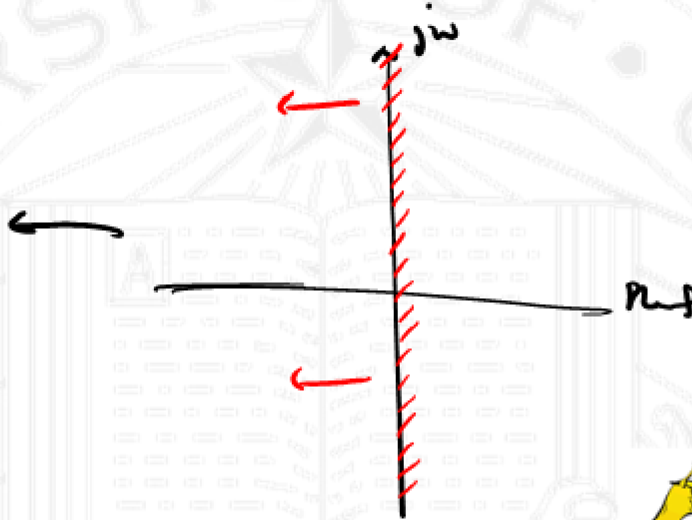
if eig(A) are $\lambda_1 \dots \lambda_n$ $\qquad \Phi \doteq e^{A\delta t}$

eig(Φ) are $e^{\lambda_1 \delta T} \dots e^{\lambda_n \delta T}$

if $\text{Real}(\lambda_i) < 0 \longleftrightarrow \left| e^{\lambda_i \delta T} \right| < 1$



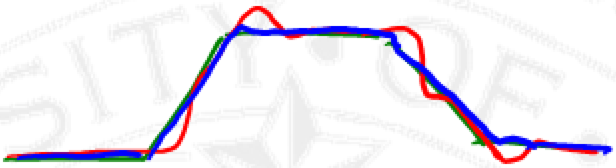
z-plane



s-plane

Extensions/Variations on discrete systems

(1) offsets : u_k and x_k, y_k are not at the same time.



(2) Multirate : u_i & x_k, y_k are not at the same ΔT.

(almost always a integer multiple of a smaller h)

Dual System (1.3)

not necessarily square

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$\longrightarrow \begin{bmatrix}\dot{x}\\ y\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}$$

$$\dot{z} = A^T z + C^T v$$
$$w = B^T z + D^T v$$
$$\longrightarrow \begin{bmatrix}\dot{z}\\ w\end{bmatrix} = \begin{bmatrix}A^T & C^T\\ B^T & D^T\end{bmatrix}\begin{bmatrix}z\\ v\end{bmatrix}$$

$$H(s) = C[sI - A]^{-1}B + D$$

$$\tilde{H}(s) = B^T(sI - A^T)^{-1}C^T + D^T \longleftarrow H(s)^T$$

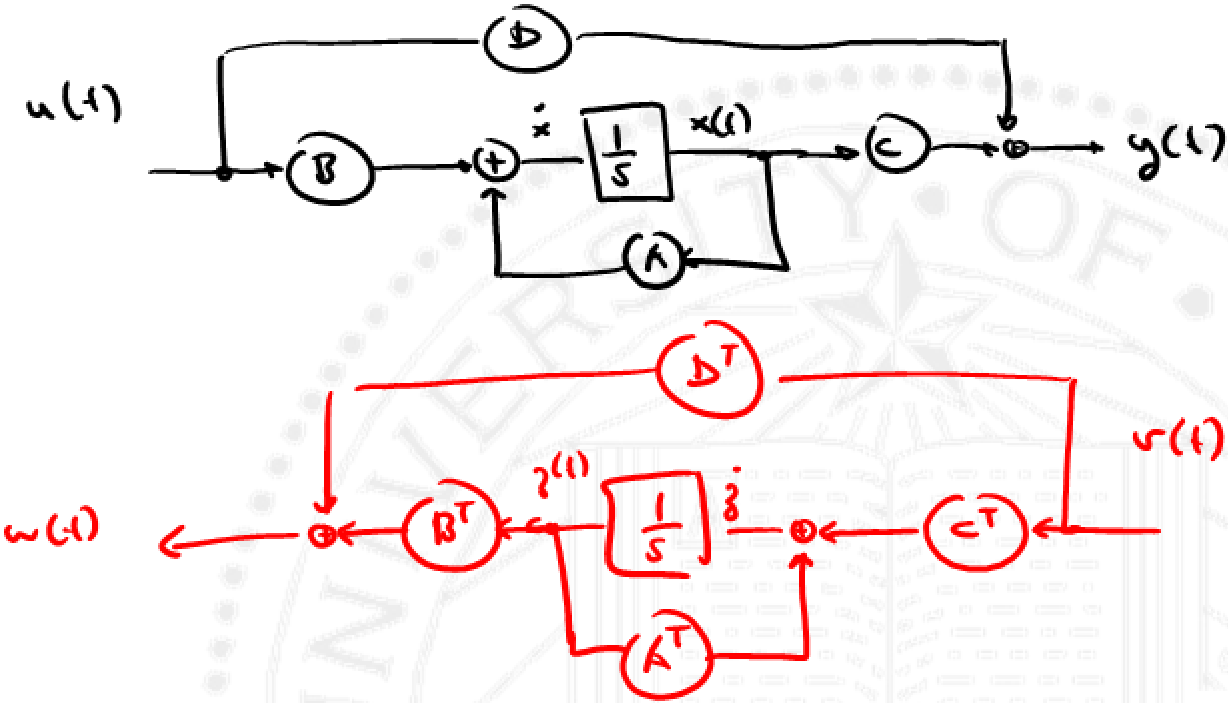
Dual System (2.3)

Single Input / Single Output (SISO) — TF is the same

eigenvalues (hence all stability properties) are also the same.

Left & right eigenvectors swap places.

Dual System (3.3)



Causality (1.3)

$$\begin{cases} x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\ y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \end{cases}$$

$t \geq 0$: current state $x(t)$ and the output $y(t)$

depends <u>only</u> on past input $u(\tau)$ $\tau \leq t$

$x(t)$ <u>CANNOT</u> be affected by <u>FUTURE INPUTS</u>

mapping from input to state is <u>CAUSAL</u> with a fixed x_0

$$x(t) - f(x_0, u(0,T))$$

Causality (2.3)

fixed final state $x(T)$ for $t \leq T$

$$\longrightarrow x(t) = e^{A(t-T)}x(T) + \int_T^t e^{A(t-\tau)}Bu(\tau)\,d\tau .$$

$x(t)$ depends only on $X(T)$, $u(t,T)$.

final state and all future inputs tell me what the current state and output are.

<u>ANTI-CAUSAL</u>

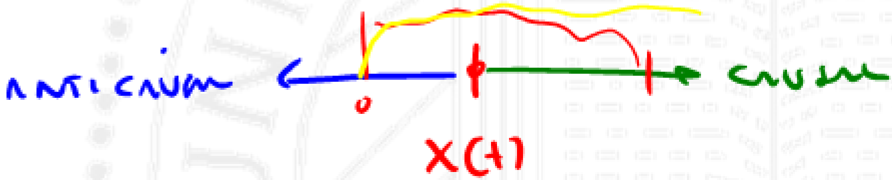
Causality (3.3)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

causality $\longrightarrow$ Fixed $x(0)$

anti-causality $\longrightarrow$ Fixed $x(T)$



Idea of State

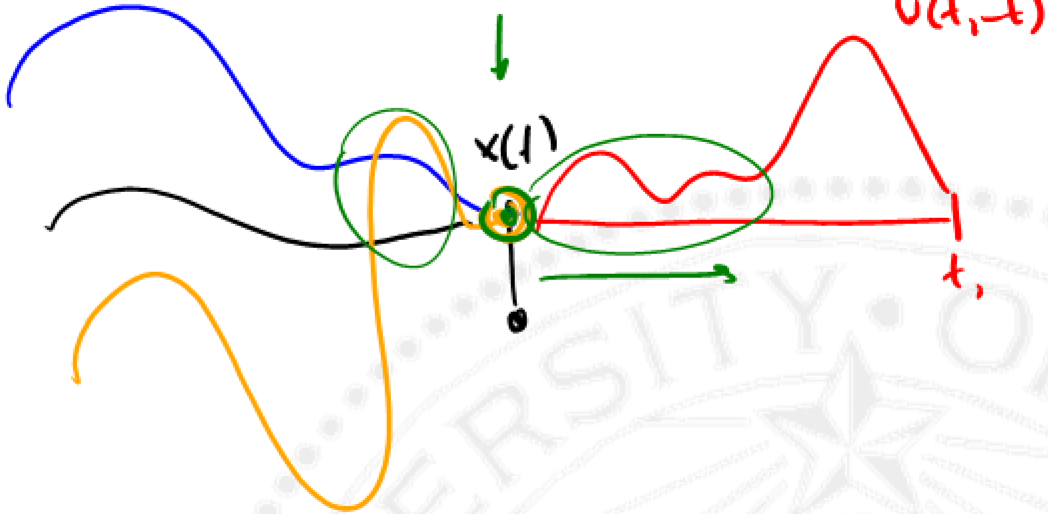
$x(t)$ is called the "state" of the system at time $= t$.

- future outputs depend ONLY on current state & future inputs

- future outputs depends on past inputs only through the current state

- State summarizes effect of all past inputs on the future outputs

- State is a bridge between past inputs & future outputs!


x(t)

Change of Coordinates

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$x \in \mathbb{R}^n \qquad \tilde{x} \in \mathbb{R}^n \qquad x = T\tilde{x} \qquad \tilde{x} = T^{-1}x$

$$\dot{(T\tilde{x})} = AT\tilde{x} + Bu \qquad \dot{(T\tilde{x})} = T\dot{\tilde{x}}$$

$$y = CT\tilde{x} + Du$$

$$\left[\begin{aligned} \dot{\tilde{x}} &= T^{-1}AT\tilde{x} + T^{-1}Bu \\ y &= CT\tilde{x} + Du \end{aligned}\right]$$

$$\begin{bmatrix} \dot{x} \\ \hdashline y \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ \hdashline u \end{bmatrix}$$

$$\begin{bmatrix} \dot{\tilde{x}} \\ \hdashline y \end{bmatrix} = \left[\begin{array}{c|c} \overbrace{T^{-1}AT}^{\tilde{A}} & \overbrace{T^{-1}B}^{\tilde{B}} \\ \hline \underbrace{CT}_{\tilde{C}} & D \end{array}\right] \begin{bmatrix} \tilde{x} \\ \hdashline u \end{bmatrix}$$

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + D = CT(sT^{-1}T - T^{-1}AT)^{-1}T^{-1}B + D$$

$$CTT^{-1}(sI - A)^{-1}TT^{-1}B + D = C(sI - A)^{-1}B + D.$$

Standard Forms for LDS (1.3)

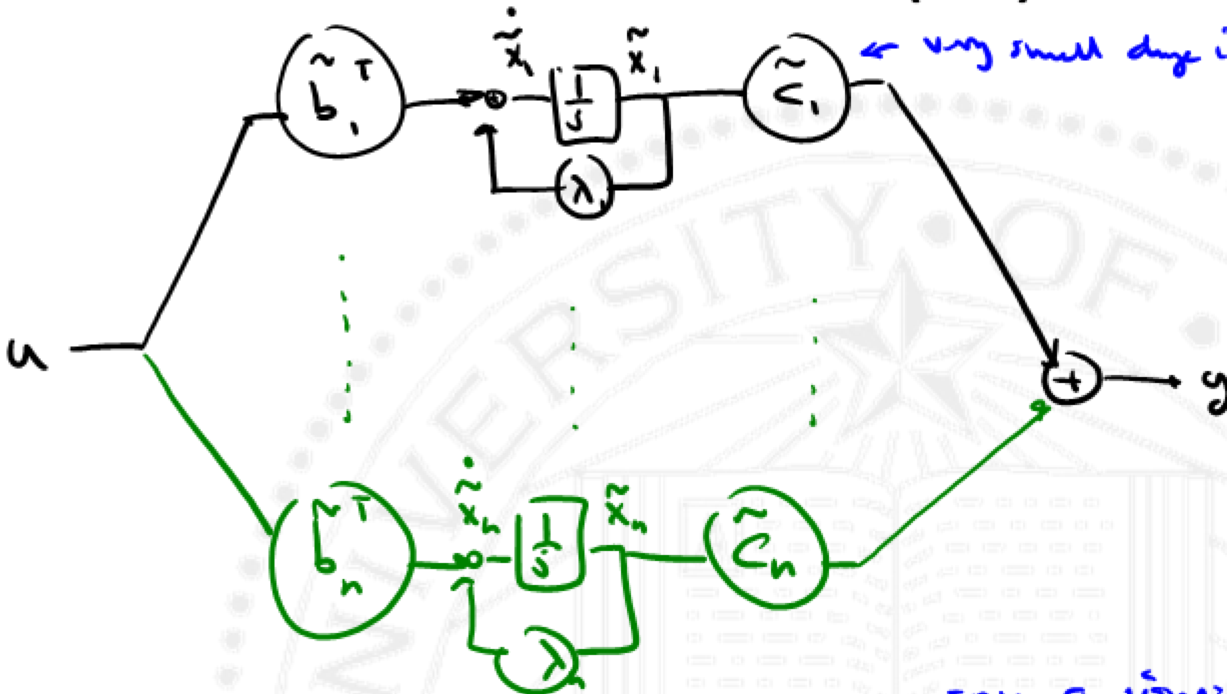
Diagonal form: find T such that

$$T^{-1}AT = \mathrm{diag}(\lambda_1 \ldots \lambda_n) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \Lambda$$

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix} \qquad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$

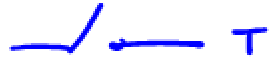
$$\boxed{\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u} \qquad \boxed{y = \sum_{i=1}^{n} \tilde{c}_i \tilde{x}_i}$$

Standard Forms for LDS (2.3)



← very small change in $\tilde{c}$

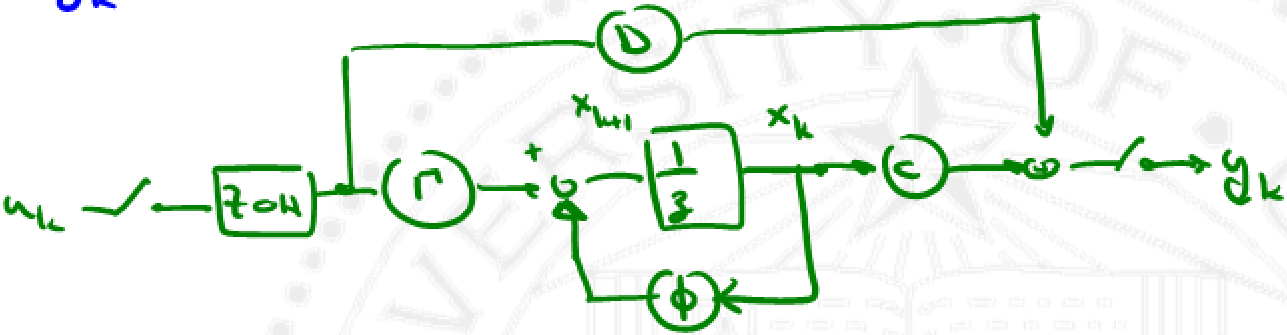
serial form / cascade form VERY SENSITIVE



Standard Forms for LDS (3.3)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = C x_k + D u_k$$



$$\frac{1}{s} \longleftrightarrow \frac{1}{z}$$

z — unit delay

Discrete-Time Systems (1.3)

$$x_1 = Ax_0 + Bu_0$$

$$x_2 = Ax_1 + Bu_1 = A(Ax_0 + Bu_0) + Bu_1 = A^2 x_0 + ABu_0 + Bu_1$$

$$x_3 = Ax_2 + Bu_2 = A(A^2 x_0 + ABu_0 + Bu_1) + Bu_2 = A^3 x_0 + A^2 Bu_0 + ABu_1 + Bu_2$$

for $k \in \mathbb{Z}_+$

$$x_k = A^k x_0 + \sum_{i=1}^{k-1} A^{(k-1-i)} B u_i$$

$$y_k = CA^k x_0 + h * u$$

← discrete time convolution

Discrete-Time Systems (2.3)

$$h_k = \begin{cases} D & k=0 \\ CA^{k-1}B & k>0 \end{cases}$$

$D, CB, CAB, CA^2B, CA^3B, \ldots$ ← markov params

$$x_{k+10} = A^{10} x_k \longleftrightarrow x(t+10) = e^{A\cdot 10} x(t)$$

$$x_{k-10} = A^{-10} x_k \qquad x(t-10) = e^{-A10} x(t)$$

↑ might not exist

Discrete-Time Systems (3.3)

FIR

$$x_k = h_1 x_{k-1} \cdots h_n x_{k-n} + u_k$$

$$x_{k+1} = \begin{bmatrix} 0 & h_1 & & \\ & \ddots & \ddots & \\ & & \ddots & h_n \\ & & & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u_k$$

$$\begin{bmatrix} x_k \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \end{bmatrix}$$

m.a.c.

DSP

Z Transform (1.3)

$w \in \mathbb{R}^{p \times q}$ sequence

$w : \mathbb{Z}_+ \to \mathbb{R}^{p \times q}$

$W = Z(w) \qquad D \subseteq \mathbb{C} \to \mathbb{C}^{p \times q}$

$$W(z) \triangleq \sum_{k=0}^{\infty} z^{-k} w(t)$$

$v(t) = w(t+1) \qquad t = 0, 1, \ldots$

$$V(z) = \sum_{t=0}^{\infty} z^{-t} w(t+1) = z \sum_{t=1}^{\infty} z^{-t} w(t) = z W(z) - z w(0)$$

Z Transform (2.3)

$Z\{v_k = w_{k+1}\}$

$$V(z) = zW(z) - zw_0 \longleftrightarrow \mathcal{L}\{\dot{x}\} = sX - x_0$$