$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

input

Linear Dynamical Systems with Inputs and Outputs

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Winter 2016

Linear Dynamical Systems with Inputs and Outputs

- Inputs and Outputs: Interpretations
- Transfer Matrix
- Impulse and Step Matrices
- Examples

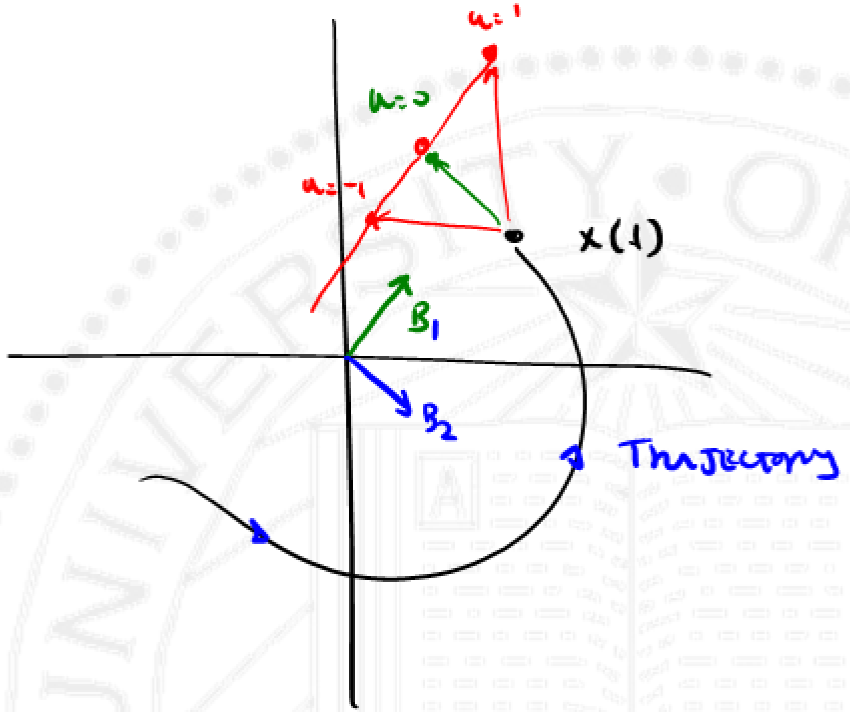
Inputs and Outputs (1.3)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Ax — drift term
homogeneous term
unforced term
ballistic term

Bu is called the input term
driving term
forced term

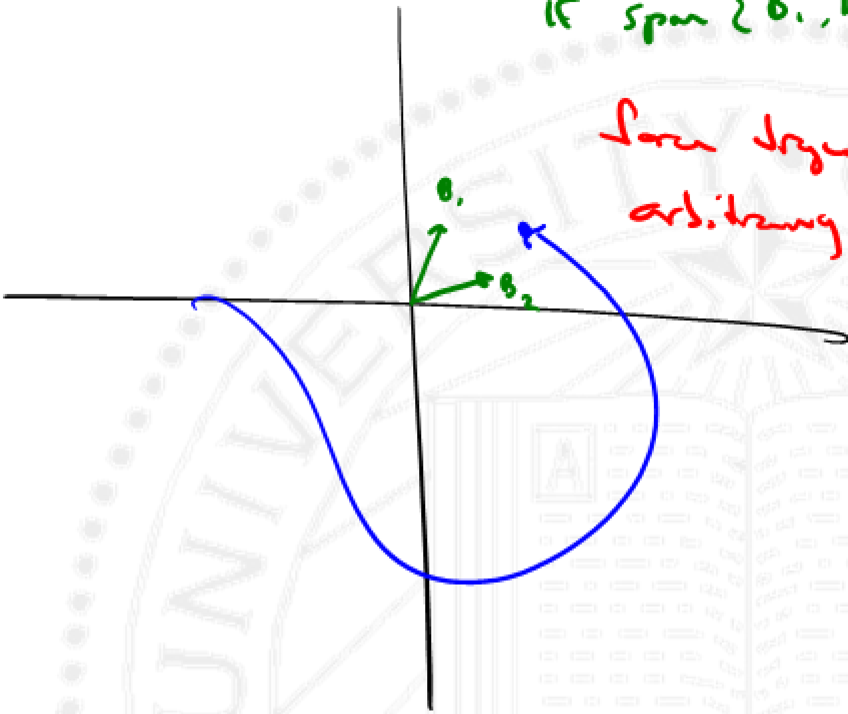
Inputs and Outputs (2.3)



Inputs and Outputs (3.3)

if span$\{b_1, b_2\} = \mathbb{R}^2$

force trajectory to any arbitrary location.



Interpretations (1.2)

$$\dot{x} = Ax + b_1 u_1 + b_2 u_2 + \dots + b_m u_m \qquad B = [b_1\ b_2\ \dots\ b_m]$$

State derivative is the sum of Autonomous Term (Ax)
and one term per input ($b_i u_i$)

Each input u_i gives another degree of freedom
for $\dot{x}$ (Assuming columns of B are independent).

Interpretations (2.2)

$\dot{x} = Ax + Bu$ i^{th} row of A

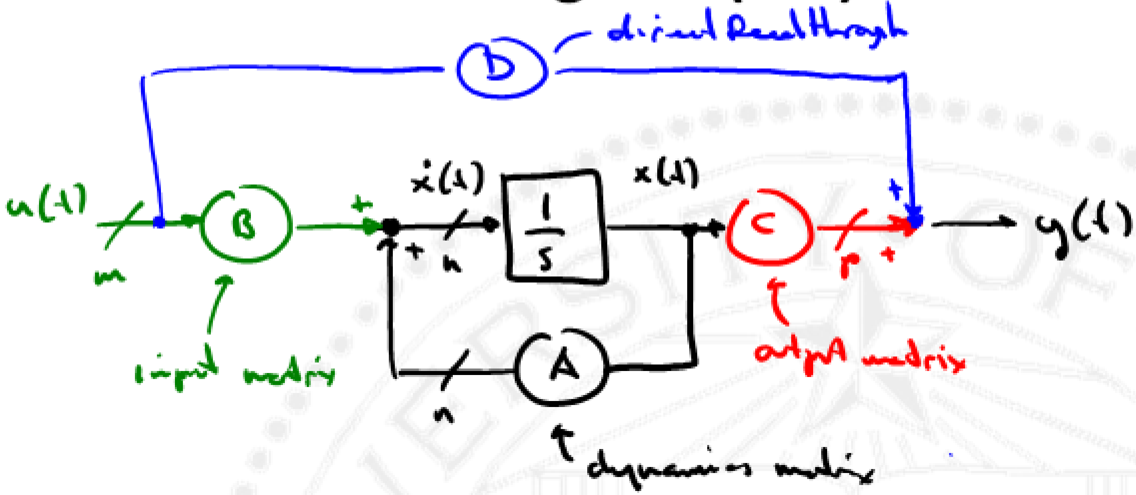
$x_i = \tilde{a}_i^T x + \tilde{b}_i^T u$

i^{th} row of B

i^{th} state derivative is a <u>LINEAR FUNCTION</u> of the state x and the input u.

Range (B) additional velocity I can hit w/ u.

Block Diagram (1.3)



A_{ij} gain from state x_j to integrator i

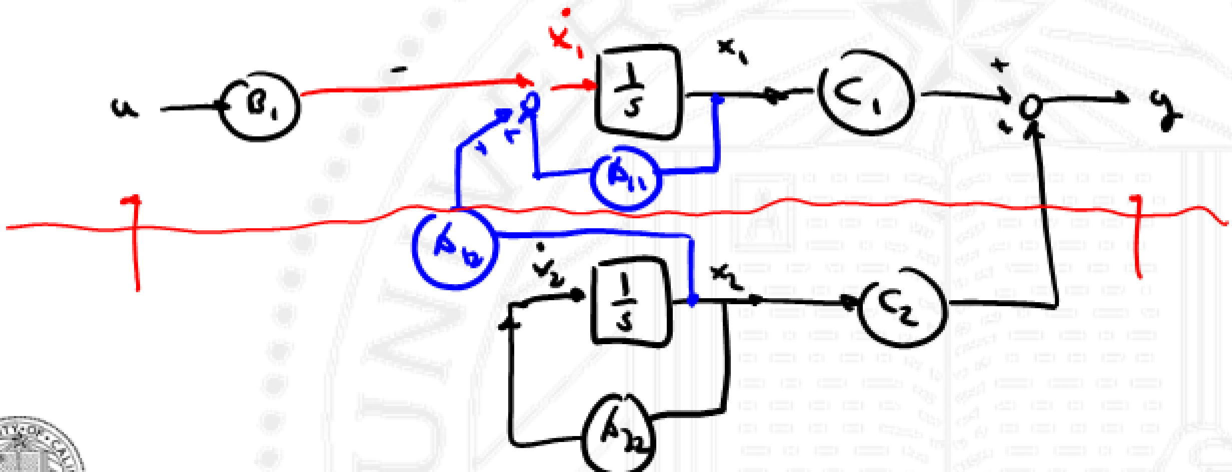
B_{ij} gain from input u_j to integrator i

C_{ij} gain from state x_j to output y_i

D_{ij} gain from input u_j to output y_i

Block Diagram (2.3)

$$\dot{\begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix}} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ \cdots \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} C_1 \\ \cdots \\ C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix}$$



Transfer Matrix (1.3)

$$\mathcal{L}\{\dot{x} = Ax + Bu\} = sX(s) - x_0 = AX(s) + \underline{BU(s)}$$

$$X(s) = \underbrace{[sI - A]^{-1}}_{\text{we know}} x_0 + \underbrace{[sI - A]^{-1} B\, U(s)}_{\text{product of two Laplace transforms}}$$

$$x(t) = e^{At} x_0 + e^{At} B * u(t) \longleftarrow \text{convolution.}$$

$$\boxed{x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}$$

Transfer Matrix (2.3)

$e^{At}x_0$ – unforced, autonomous, homogeneous, natural resp.

$e^{At}B$ – input to state impulse matrix $(u_i = \delta_i)$

$(sI-A)^{-1}B$ – called the input to state transfer matrix

transfer function matrix.

$\mathcal{L}\{y = Cx + Du\} \rightarrow Y(s) = CX(s) + DU(s)$

Transfer Matrix (3.3)

$$Y(s) = C[sI-A]^{-1}x_0 + \left(C[sI-A]^{-1}B + D\right)U(s)$$

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{initial condition response}} + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$H(s) \triangleq C[sI-A]^{-1}B + D$$ transfer function matrix

$$h(t) = \left[Ce^{At}B + D\right]\delta(t)$$ ← impulse response or impulse response matrix

$\delta(t)$: dirac delta

Impulse Matrix (1.3)

$x(0) = 0.$ $\quad h(t) = \left[Ce^{At}B + D\right]\delta(t)$

$Y(s) = H(s)U(s) \longrightarrow y(t) = h(t) * u(t)$

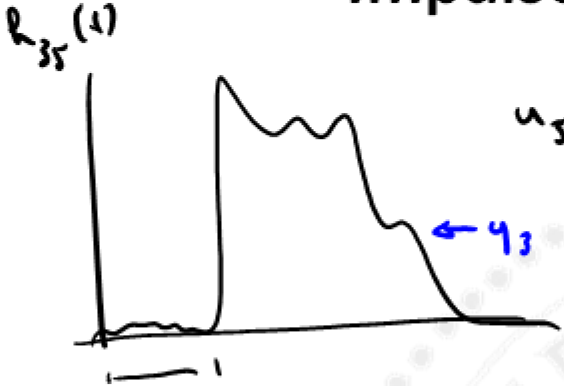
H_{ij} transfer function from input u_j to output y_i

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t-\tau)\,u(\tau)\,d\tau .$$

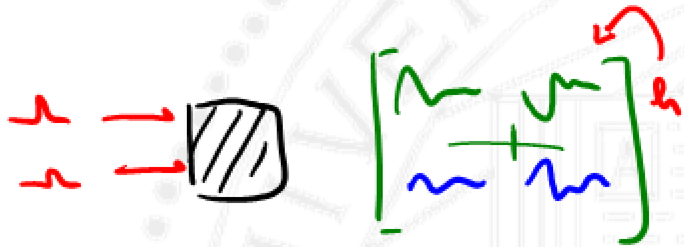
h_{ij} are the impulse response from j^{th} input to i^{th} output

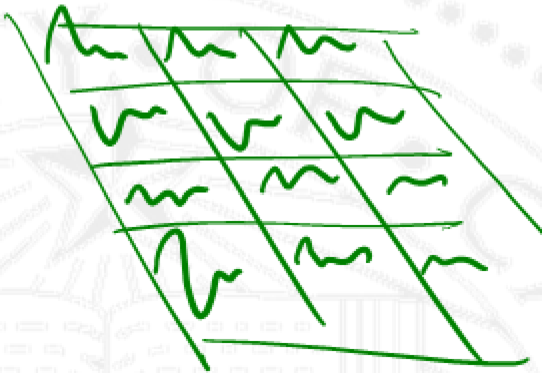
$\underline{h_{ij}(t)}$ gives $y_i(t)$ when $\underline{u_j = e_j\delta(t)}$

Impulse Matrix (2.3)



$u_5 \to$ $e t = \phi$





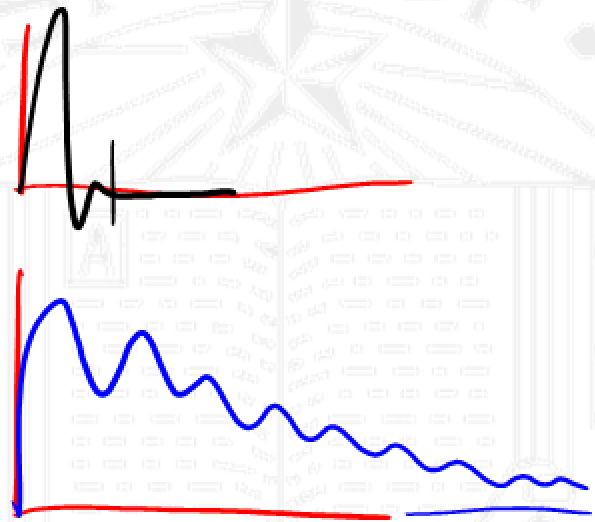
Impulse Matrix (3.3)

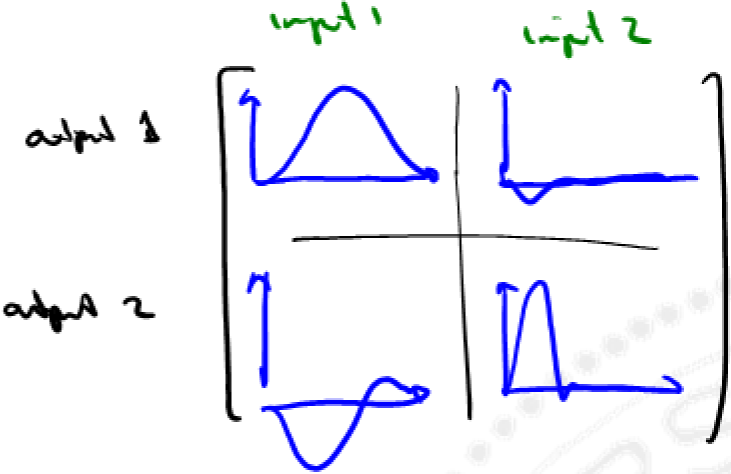
$h_{ij}(\tau)$ shows how dependent y_i is on input u_j, τ seconds ago.

i = output

j = input

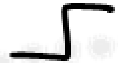
τ = time lag.




input 1
input 2
output 1
output 2

Step Matrix (1.3)

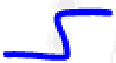
$s(t) = \int_0^t h(\tau)\, d\tau$

input  $1 \ \forall t > 0$

$t=0$

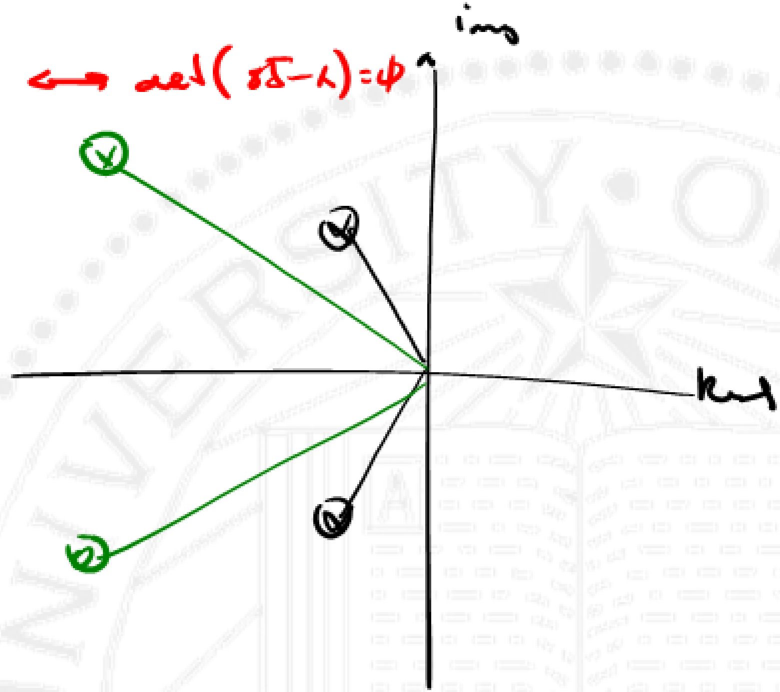
$s_{ij}(t)$ step response from j^{th} input to i^{th} output

$s_{ij}(t)$ gives $y_i(t)$ when input is e_j $\forall t \geq 0$.

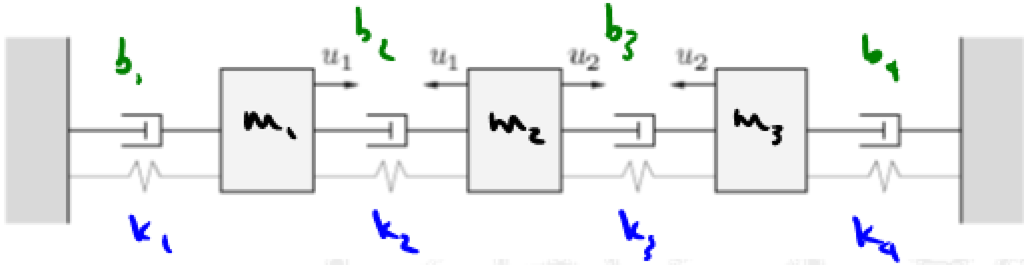


Step Matrix (2.3)

$\mathrm{eig}(A) \longleftrightarrow \det(sI - A) = 0$



Example: Mass-Spring-Damper (1.3)



$m_1 = m_2 = m_3 = 1$ kg.

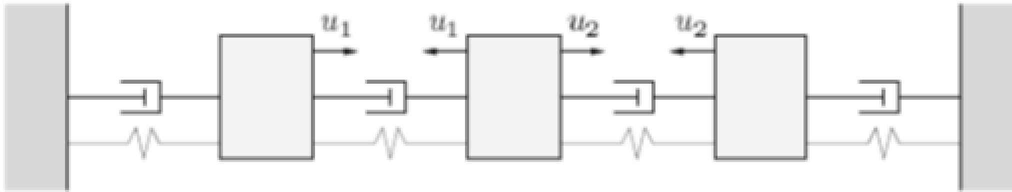
$k_1 = k_2 = k_3 = k_4 = 1$ N/m

$b_1 = b_2 = b_3 = b_4 = 1$ N/m/s

$y \in \mathbb{R}^3$ $\qquad \underline{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

$$\dot{\underline{x}} = \left[\begin{array}{c|c} 0 & I \\ \hline \cdots & \cdots \end{array}\right] \underline{x} + \begin{bmatrix} 0 \\ \square \end{bmatrix} \underline{u} \longrightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Example: Mass-Spring-Damper (2.3)



$$\dot{\underline{x}} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

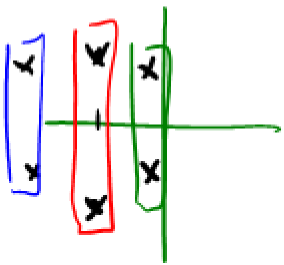
$$y = \left[\begin{array}{c|c} I & 0 \end{array}\right] \underline{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \end{matrix}$$

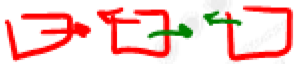
Example: Mass-Spring-Damper (3.3)

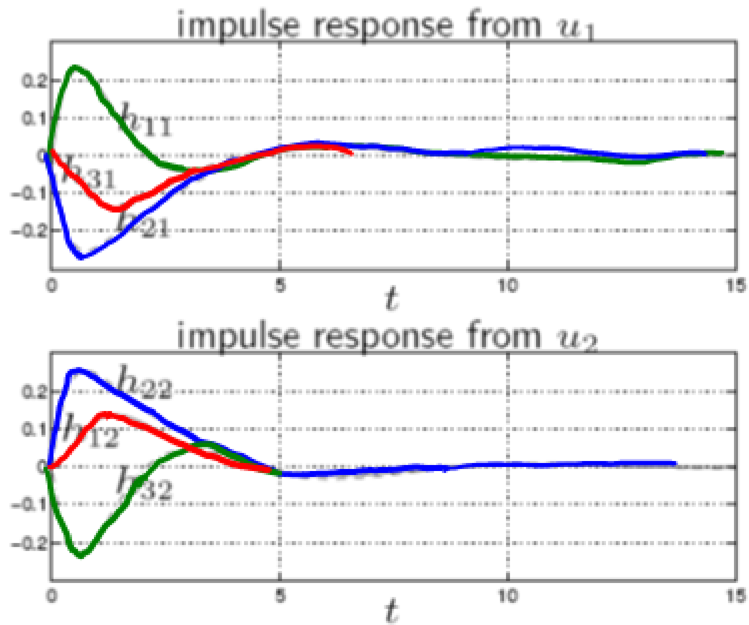
$\operatorname{eig}(A) = \boxed{-1.71 \pm .71j}$ ← osc

$\boxed{-1 \pm 1j}$

$\boxed{-0.29 \pm .71j}$





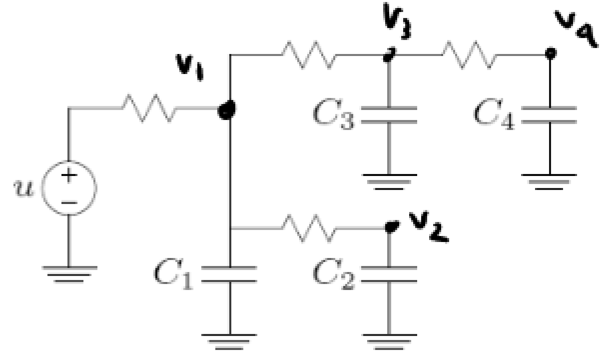


h_{ij} $\quad H \in \mathbb{R}^{3\times 2}$

u_1 affects third mass, less than other two

u_2 affects first mass less than other two.

Example: Interconnect Circuit (1.3)



$t=0 \rightarrow$

$u(t) \in \mathbb{R}$ input drive voltage

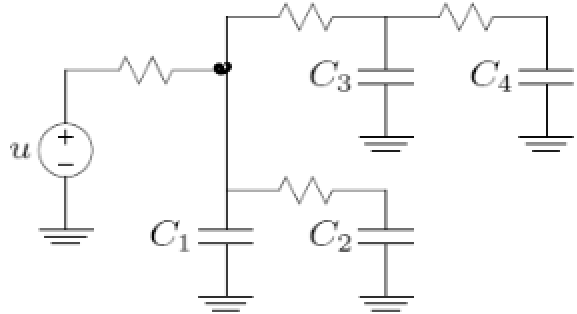
GATE VOLTAGE SOURCE

x_i voltage across capacitor C_i

output then $y = x$.   1V   1Ω   1F

Example: Interconnect Circuit (2.3)

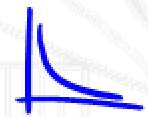
$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

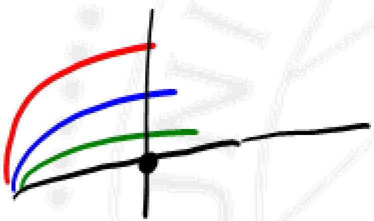


$$y = I x$$

$$\mathrm{eig}(A) = \begin{bmatrix} -0.17 \\ -0.66 \\ -2.21 \\ -3.96 \end{bmatrix}$$

$$e^{\lambda t}$$





Example: Interconnect Circuit (3.3)

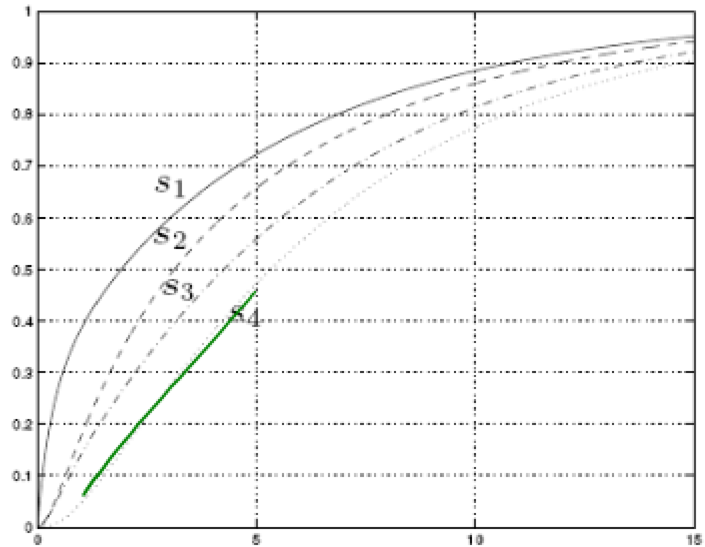
$s(t) \in \mathbb{R}^{4\times 1}$

shortest delay is to x_1

x_1 is fastest

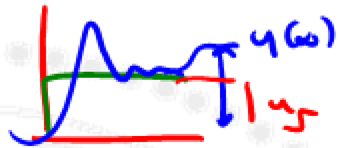
longest delay is to x_4

desired equilibrium -0.17



DC or Static Gain Matrix (1.3)

Transfer function Matrix $\Big|_{s=0}$.



$$H(s)\Big\|_{s\to 0} = C[sI-A]^{-1}B+D\Big|_{s=0} \in \mathbb{R}^{m\times p}$$

$$\overset{0}{\cancel{\dot{x}}} = Ax+Bu \qquad 0 = Ax+Bu \qquad x = -A^{-1}Bu$$

$$y = Cx+Du \qquad y = -[CA^{-1}B+D]u$$

$$\frac{y}{u}\Big|_{t=\infty} \longleftrightarrow \underline{H(0)}.$$

DC or Static Gain Matrix (2.3)

$y = H(0)u$

step response

$H(0) = \int_0^\infty h(t)\,dt = \lim_{t \to \infty} s(t)$

MUST BE STABLE

$\frac{1}{(s-1)} \to \cancel{h(t) = -1}$

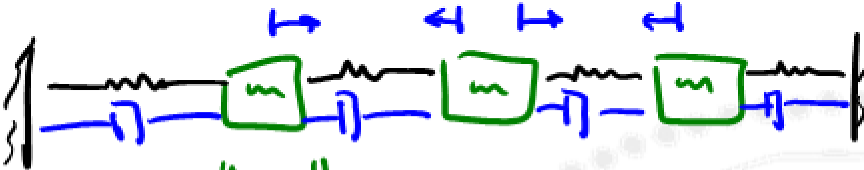
e^t

$u(t) \to u_\infty \in \mathbb{R}^m$

$y(t) \to y_\infty \in \mathbb{R}^p$

$H(s) = \int_0^\infty e^{-st} h(t)\,dt \to s(t) = \int_0^t h(\tau)\,d\tau$

DC or Static Gain Matrix (3.3)



$$H(0) = \begin{array}{c c} & \begin{array}{cc} u_1 & u_2 \end{array} \\ & \left[\begin{array}{cc} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{array}\right] \end{array} \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}$$

$$u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

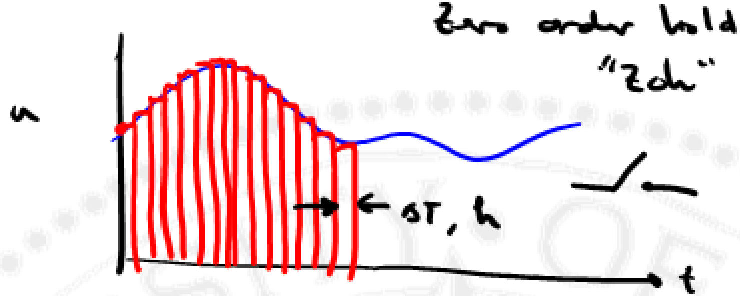
$$y_0 = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$H(0) = C(-A)^{-1}B + \cancel{D}^{\,0} \longrightarrow \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$$

Discretization with Piecewise Constant Inputs (1.3)

$\dot{x} = Ax + Bu$

$y = Cx + Du$



$u_d : \mathbb{Z}_+ \to \mathbb{R}^m$ is an input sequence

$u(t) = u_d(k)$ for $kh \le t \le (k+1)h$

$x_d(k) = x(kh)$ $\quad y_d(k) = y(kh)$ $\quad k = 0, 1, \ldots$

$h > 0$ sample interval (ΔT)

u is piecewise constant (zero order hold)

Discretization with Piecewise Constant Inputs (2.3)

$$x_d(k+1) = x((k+1)h)$$

constant

$$= e^{Ah}x(kh) + \int_0^h e^{A\tau} B\,\overbrace{u((k+1)h-\tau)}\,d\tau$$

$$= \underbrace{e^{Ah}}_{A_d} x_d(k) + \underbrace{\left(\int_0^h e^{A\tau} B\,d\tau\right)}_{B_d} u_d(k)$$

$$\dot{x} = Ax + Bu$$

NOT AN APPROX

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k)$$

$$y_d(k) = C x_d(k) + D u_d(k)$$

Discretization with Piecewise Constant Inputs (3.3)

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Gamma \underline{u}_k$$

$$\Phi = e^{A \Delta T}$$

$$y_k = C \underline{x}_k + D \underline{u}_k$$

$$\Gamma = \left[\int_0^{\Delta T} e^{At}\, dt\right] B$$

if A is invertible:

$$\int_0^{\Delta T} e^{At}\, dt\, B = A^{-1}\left(e^{A\Delta T} - I\right) B$$

STABILITY is PRESERVED