Jordan Canonical Form

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Jordan Canonical Form

- Jordan canonical form
- Generalized modes
- Cayley-Hamilton theorem

Jordan Canonical Form (1.4)

What if $A \in \mathbb{R}^{n \times n}$ cannot be diagonalized?

<u>any</u> matrix $A \in \mathbb{R}^{n \times n}$ can be put into <u>JORDAN</u> <u>Canonical</u> <u>Form</u> by a similarity transformation

$$T^{-1} A T = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

block diagonal form

$[\lambda_1]$ 1×1

$\begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}$ 2×2

$\begin{bmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_3 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ 3×3

Jordan Canonical Form (2.4)

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i} \qquad n = \sum_{i=1}^{q} n_i$$

Jordan Block of size n_i with eigenvalue λ_i

- $\to$ J_i is upper bi-diagonal
- J is diagonal "special case" n jordan block of $n_i = 1$.
- J is <u>unique</u> up to permutations of blocks.

Jordan Canonical Form (3.4)

Can have multiple Jordan Blocks w/ same λ_i

$\lambda = (1,1,1,1)$

$$J = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad 4 @ 1\times1$$

$\underline{N(\lambda I - J)}$

$$J = \left[\begin{array}{cc|cc} 1 & 1 & & \\ & 1 & & \\ \hline & & 1 & 1 \\ & & & 1 \end{array}\right] \quad 2 @ 2\times2$$

$$J = \left[\begin{array}{cc|c|c} 1 & 1 & & \\ & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right] \quad \begin{array}{l} 1 @ 2\times2 \\ 2 @ 1\times1 \end{array}$$

$$J = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \quad 1 @ 4\times4$$

$$J = \left[\begin{array}{ccc|c} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ \hline & & & 1 \end{array}\right] \quad \begin{array}{l} 1 @ 3\times3 \\ 1 @ 1\times1 \end{array}$$

$$T^{-1}AT = J \quad \therefore \quad A = TJT^{-1} \qquad I = TT^{-1}$$

Jordan Canonical Form (4.4)

$$\mathcal{X}(s) = \det(sI - A) = \det\left(T(sI-J)T^{-1}\right) = \det(T)\det(sI-J)\det(T^{-1})$$

$$= (s-\lambda_1)^{n_1}(s-\lambda_2)^{n_2}\cdots(s-\lambda_q)^{n_q}$$

distinct eigenvalues $\rightarrow n_i = 1 \rightarrow A$ is fully diagonalizable

$\dim \overset{\text{null}}{N}(\lambda I - A)$ the number of Jordan Blocks with eigenvalue λ

$$\dim N(\lambda I - A)^k = \sum_{\lambda_i = \lambda} \min(k, n_i)$$

$$\left[I - \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & & -1 \\ & & & 0 \end{bmatrix}^k \leftarrow \text{shift matrix}$$

$$(\lambda_1 I - J_1) = \begin{bmatrix} 0 & -1 & 0 \\ & \ddots & -1 \\ & & 0 \end{bmatrix}$$

$$(\lambda_1 I - J_1)^2 = \begin{bmatrix} 0 & 0 & 1 \\ & \ddots & 0 \\ & & 0 \end{bmatrix}$$

$$(\lambda_1 I - J_1)^3 = \begin{bmatrix} \phi \end{bmatrix}$$

$$(\lambda_i I - J_i)^k = \begin{bmatrix} (\lambda_i-\lambda_j)^k & -k(\lambda_i-\lambda_j)^{k-1} & -\frac{k(k-1)(\lambda_i-\lambda_j)^{k-2}}{2} \\ 0 & (\lambda_i-\lambda_j)^k & -k(\lambda_i-\lambda_j)^{k-1} \\ 0 & 0 & (\lambda_i-\lambda_j)^k \end{bmatrix}$$

$$\left(1 I - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{rank } 2.$$

2 jordan block.

Generalized Eigenvectors (1.3)

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

$$T = [T_1 \; T_2 \; \cdots \; T_q]$$ where $T_i \in \mathbb{C}^{n \times n_i}$ columns of T associated w/ Jordan Block J_i

$$AT_i = T_i J_i$$

$$T_i = [v_{i_1} \; v_{i_2} \; \cdots \; v_{i_{n_i}}]$$

$$\boxed{Av_{i_1} = \lambda_i v_{i_1}}$$

normal eigenvalue and eigenvector.

Generalized Eigenvectors (2.3)

The first column of each T_i is an eigenvector associated with eigenvalue λ_i

$$\boxed{A v_{i1} = \lambda_i v_{i1}}$$

for $j = 2 \ldots n_i$

$$\boxed{A v_{ij} = v_{i(j-1)} + \lambda_i v_{ij}}$$

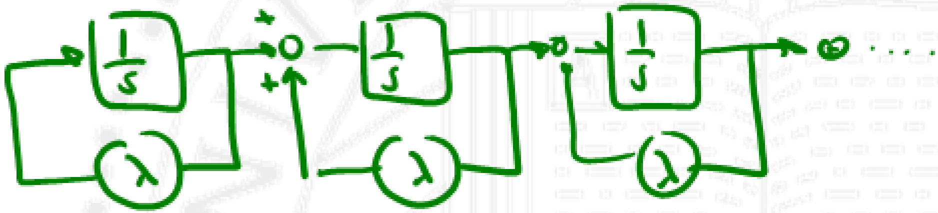
a generalized eigenvector.

Jordan Form LDS (1.4)

$\dot{x} = Ax \qquad x = T\tilde{x}$

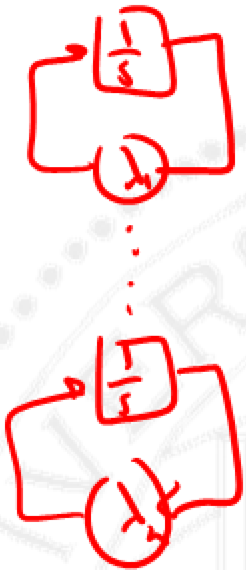
$\dot{\tilde{x}} = J\tilde{x} \qquad J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$

$\dot{\tilde{x}}_i = J_i \tilde{x}_i$ ← independent subsystems.

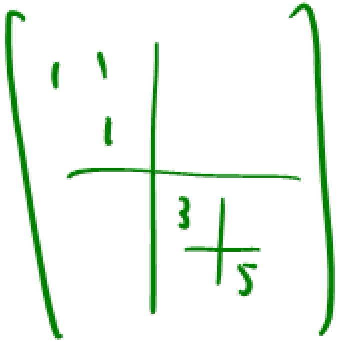


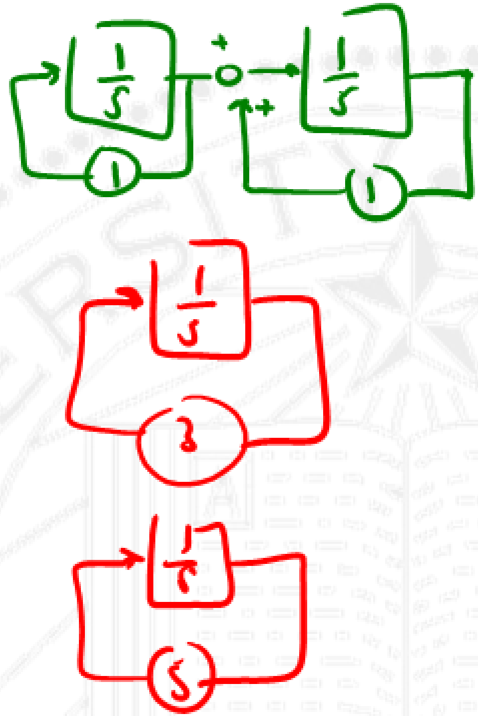
Jordan chains

Jordan Form LDS (2.4)



Jordan Form LDS (3.4)





Jordan Form LDS (4.4)

$$\dot{x} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} x$$

$eig(A) = (0,0,0,0)$

solution: constant (t^0)

t

t^2

t^3

Resolvent and Exponential of Jordan Block (1.3)

$$(sI - J_\lambda)^{-1} = \begin{bmatrix} s-\lambda & -1 & & \\ & s-\lambda & \ddots & \\ & & \ddots & -1 \\ & & & s-\lambda \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (s-\lambda)^{-1} & (s-\lambda)^{-2} & \cdots & (s-\lambda)^{-k} \\ & (s-\lambda)^{-1} & & \vdots \\ & & \ddots & (s-\lambda)^{-2} \\ & & & (s-\lambda)^{-1} \end{bmatrix}$$

poles up to order k

Resolvent and Exponential of Jordan Block (2.3)

$$(sI - J_\lambda)^{-1} = \frac{1}{s-\lambda} I + \frac{1}{(s-\lambda)^2} F_1 + \dots + \frac{1}{(s-\lambda)^k} F_{k-1}$$

$$F_i = \begin{bmatrix} 0 & & 1 & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

1s along i^{th} super diagonal

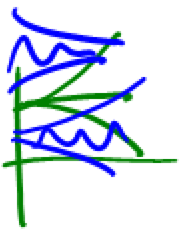
$$e^{J_\lambda t} = e^{\lambda t}\left[I + tF_1 + \frac{t^2}{2!} F_2 + \dots \frac{t^{(k-1)}}{(k-1)!} F_{k-1} \right]$$

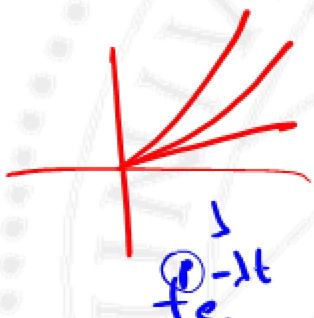
Resolvent and Exponential of Jordan Block (3.3)

$$e^{J_\lambda t} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & \ddots & \ddots & \ddots & \frac{t^2}{2} \\ & & \ddots & \ddots & t \\ & & & & 1 \end{bmatrix}$$

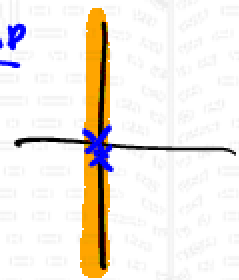
$e^{J_\lambda t} \leftarrow$ jordan blocks

- Repeated poles in the resolvent
- terms in the form $t^r e^{\lambda t}$ in e^{At}.

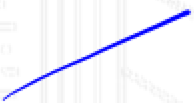




LHP



t



Generalized Modes (1.3)

$\dot{x} = Ax \qquad x(0) = a_1 v_{i_1} + a_2 v_{i_2} + \dots + a_{n_i} v_{i_{n_i}} = \underline{T_i a}$

$\underline{x(t)} = T e^{Jt} \tilde{x}(0) = \underline{T_i e^{J_i t} a} \qquad \tilde{x}_i(0)$

- Trajectory of $x(t)$ stay in the span $\{v_{i_1} \dots v_{i_{n_i}}\}$

- Coefficients have the form of $p(t) e^{\lambda t}$

  $p(t)$ is a polynomial in t

- Solution — $p(t) e^{\lambda t}$ — "generalized modes"

Generalized Modes (2.3)

For a generic $x(0)$

$$x(t) = e^{At}x(0) = Te^{Jt}T^{-1}x(0) = \sum_{i=1}^{q} T_i e^{J_i t}(S_i^T x(0))$$

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix}$$

S_i^T are the block inverse of the T_i

$$T = \begin{bmatrix} T_1 & \cdots & T_q \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix}$$

Generalized Modes (3.3)

All solutions of $\dot{x} = Ax$ are <u>linear</u> combinations of the generalized modes

(1) Not <u>all</u> matrices are diagonalizable

(2) Distinct eigenvalues $\longrightarrow$ it is diagonalizable $T^{-1}AT = \Lambda$

(3) All matrices can be put into <u>JORDAN FORM</u>

(4) multiple poles in the resultant $t^{p}e^{\lambda t}$ response.

Cayley-Hamilton Theorem (1.4)

$p(s) = a_0 + a_1 s + a_2 s^2 + \dots + a_k s^k$ polynomial

$p(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_k A^k$ matrix overload

Cayley Hamilton

for any $A \in \mathbb{R}^{n \times n}$ $\quad \chi(A) = 0 \quad \chi(s) = \det(sI - A)$

$\chi(A) = \det(AI - A) = \cancel{\det(A - A)} = \det(0) = 0.$

wrong proof

Cayley-Hamilton Theorem (2.4)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \chi(\lambda) = \lambda^2 - 5\lambda - 2 = \det(\lambda I - A)$$

$$\chi(A) = A^2 - 5(A) - 2I = \phi$$

$$\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Cayley-Hamilton Theorem (3.4)

for every $p \in \mathbb{Z}_+$ we have

$A^p \in \text{span}\{I, A, A^2, \ldots, A^{n-1}\}$

if A^{-1} exists, then $p \in \mathbb{Z}$

every power of A can be expressed as a linear combination of $I, A, A^2 \ldots A^{n-1}$

Cayley-Hamilton Theorem (4.4)

divide $\chi(s)$ into s^p $\rightarrow$ $s^p = q(s)\,\chi(s) + r(s)$

degree $< n$

$$r(s) = \alpha_0 + \alpha_1 s + \dots + \alpha_{n-1} s^{n-1}$$

$$A^p = q(A)\cancelto{0}{\chi(A)} + r(A) = r(A)$$

$$= \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

coef. $s^p/\chi(s)$

Proof of Cayley-Hamilton (1.3)

$p = -1$

$$\chi(A) = \underbrace{A^n + a_{n-1}A^{n-1} + \dots + a_0 I}_{} = 0$$

$= 0$ and initially

$\downarrow$

$$a_0 I = -A^n - a_{n-1}A^{n-1} - \dots - a_1 A$$

$$I = -\frac{1}{a_0}A^n - \left(\frac{a_{n-1}}{a_0}\right)A^{n-1} - \dots - \frac{a_1}{a_0}A$$

$$I = \underbrace{\left[-\frac{1}{a_0}A^{n-1} - \frac{a_{n-1}}{a_0}A^{n-2} - \dots \left(-\frac{a_1}{a_0}\right)I\right]}_{A^{-1}} A$$

Proof of Cayley-Hamilton (2.3)

$$A^{-1} = \underbrace{-\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \dots - \frac{1}{a_0} A^{n-1}}.$$

if $\underline{a_0 = 0}$ not invertible

Inverse of A is a linear combination of A^k $k = 0 \ldots n-1$

Fast method for solving $Ax = B$ — 100×100 ~ more, grows as n^3

Proof of Cayley-Hamilton (3.3)

Assume $A \in \mathbb{R}^{n \times n}$ is diagonalizable — $T^{-1}AT = \Lambda$

$$\chi(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

$$\chi(A) = \chi(T^{-1}AT) = T\underbrace{\chi(\Lambda)}_{0 \text{ need to show this}}T^{-1}$$

$$\chi(\Lambda) = (\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)$$

$$\left(\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} - \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{bmatrix}\right)\left(\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} - \begin{bmatrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{bmatrix}\right)\cdots\left(\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} - \begin{bmatrix}\lambda_n & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\right)$$

$$\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} - \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{bmatrix} \right) = \begin{bmatrix} 0 & & & \\ & \lambda_2-\lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n-\lambda_1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_2-\lambda_1 & & & \\ & 0 & & \\ & & \lambda_2-\lambda_3 & \\ & & & \ddots \end{bmatrix}$$

$$\begin{pmatrix} 0 & & \\ & \ddots & \\ & & \end{pmatrix} \begin{pmatrix} \ddots & & \\ & 0 & \\ & & \ddots \end{pmatrix} \cdots \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & 0 \end{pmatrix} = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$$

$$T^{-1} A T = J \qquad \chi(s) = (s-\lambda_1)^{n_1} \cdots (s-\lambda_q)^{n_q}$$

$$\chi(J_i) = 0$$

$$\chi(J_i) = (J_i - \lambda_1)^{n_1} (J_i - \lambda_2)^{n_2} (\dots) \overbrace{\underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{n_i}}_{(J_i - \lambda_i I)^{n_i}}}^{=0} (\dots)(J_i - \lambda_q)^{n_q}$$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{k} = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$$

$A \in \mathbb{R}^{n \times n}$ —— $\mathbb{R}^{n \times n}$ is itself a vector space (n^2)

$\mathrm{vec}(A) : \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $A(:)$ $\longleftrightarrow$ reshape

$A \in \mathbb{R}^{n \times n}$

$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ 1 in ith place.

$E_{ij} = e_i e_j$ $i,j = 1..n$ rank 1 matrix

$A \in \mathbb{R}^{10 \times 10}$

$A_1, A_2 \ldots A_{101} \in \mathbb{R}^{10 \times 10}$

$I, A_1, \ldots A_{101}$ <u>not</u> independent

$p(s) - p(A) \qquad A^k - I, A, \ldots, A^{n-1}$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

$A \in \mathbb{R}^{3\times 3}$

$$e^A = f(I, A, A^2)$$

//

$$\underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{\Omega}$$

$$e^{\Omega t}$$

$$I \quad \Omega \quad \Omega^2 \longrightarrow \frac{\sin \|\omega\|}{\|\omega\|}$$

$$R_{k+1} = e^{\Omega \Delta T} R_k$$

Questions?

Poles of the system $\det(sI - A) = \phi$