Eigenvectors and Diagonalization

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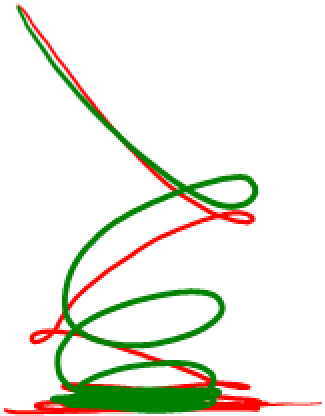
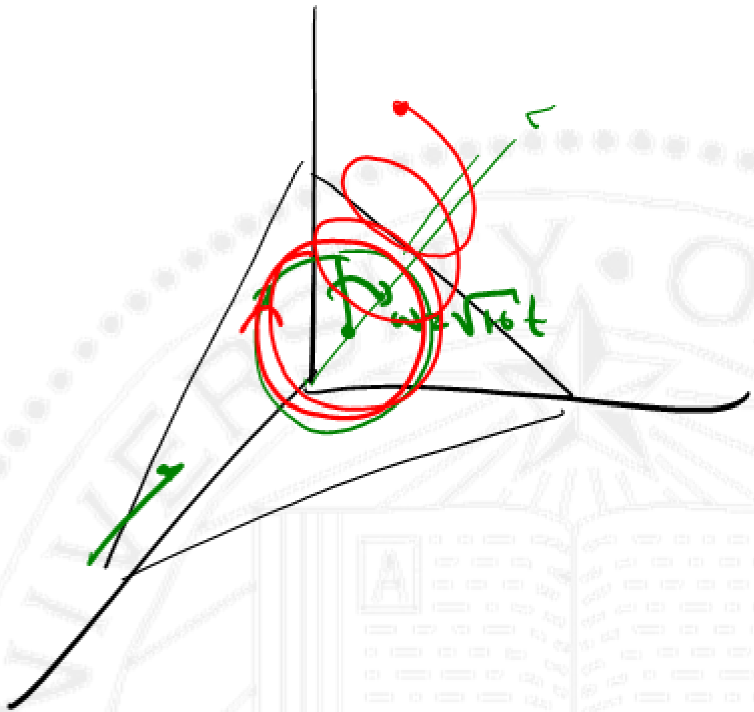




Example: Markov Chain (1.3)

$p(t+1) = P\, p(t)$

$p_i(t) = \text{prob}(z(t) = i)$ so: $\sum_{i=1}^{n} p_i(t) = 1$.

$P_{ij} \triangleq \text{prob}(z(t+1) = i \mid z(t) = j)$ so $\sum_{i=1}^{n} P_{ij} = 1$.

$\underbrace{[1 \quad 1 \quad \cdots \quad 1]}_{w^T = \mathbb{1}^T} P = [1 \quad 1 \quad \cdots \quad 1]$

$\lambda = 1$ $\quad \det(\cancel{\lambda}^{1} I - P) = \phi$

$Pv = v \quad v \neq 0$.

Example: Markov Chain (2.3)

choose to normalize v_i : $\sum_{i=1}^{n} v_i = 1$.

what is $\underline{v}$? $\lambda = 1$.

$p(0) = v$

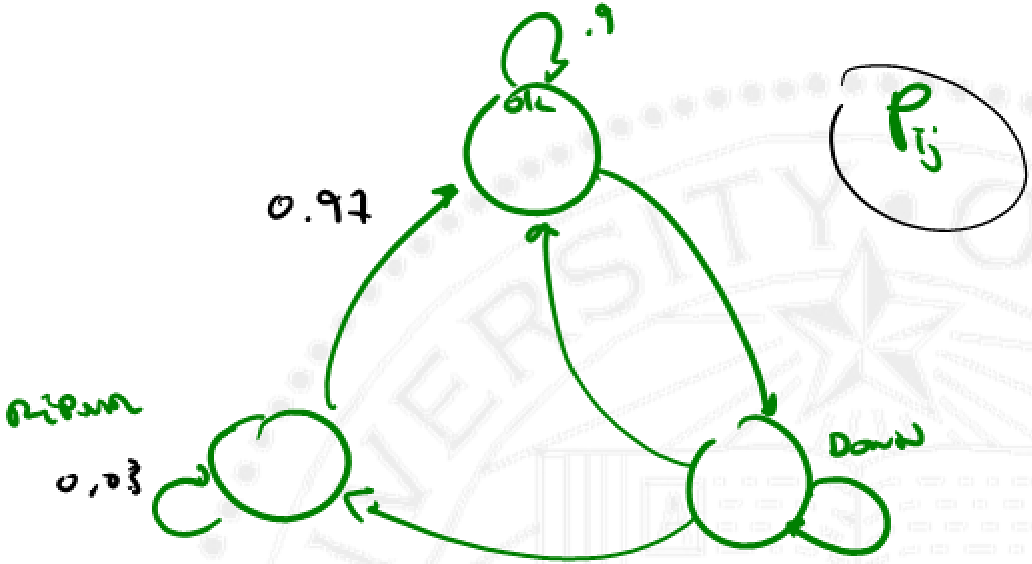
$Pv = v$

$p(t+1) = v$

$\underline{\text{if}}$ v is unique → STEADY STATE DISTRIBUTION of the markov chain

v – equilibrium distribution

Example: Markov Chain (3.3)



v =

Diagonalization (1.3)

$v_1 \ldots v_n$ linearly independent (right) eigenvectors

of $A \in \mathbb{R}^{n \times n}$

$$A v_i = \lambda_i v_i \qquad i = 1 \ldots n$$

$$A \underbrace{\begin{bmatrix} v_1 \ldots v_n \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} v_1 \ldots v_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$T^{-1} A T = T^{-1} T \Lambda$$

$$\boxed{T^{-1} A T = \Lambda}$$

~~Example: Markov Chain (1.4)~~

$$T^{-1} T = \sum_{i=1}^{j} w_i^T r_j = I$$

$$T = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

$$T T^{-1} = \sum_{i=1}^{j} \underbrace{r_i w_j^T}_{\text{diads – rank 1 matrices}} = I$$

Diagonalization (2.3)

Call A "diagonalizable"

- There exists T such that $T^{-1}AT = \Lambda$ is diagonal

- A has a set of linearly independent eigenvectors

$$T \triangleq [v_1 \cdots v_n]$$

$A \not\Rightarrow \Lambda$ "A" defective

Diagonalization (3.3)

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad \chi(s) = s^2 \qquad \lambda = (0,0)$

$$Av = 0v$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

all eigenvectors when $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ $v_1 \neq 0$.

A is defective or not diagonalizable

cannot make v_1, v_2 independent.

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \mathcal{X}(s) = \frac{1}{(s-1)^3} \qquad \lambda_i = (1, 1, 1)$$

$$Av = \lambda v$$

$v = v$ all v are eigenvectors.

$\top$

Distinct Eigenvalues

<u>Fact</u> if A has distinct eigenvalues

$\lambda_i \neq \lambda_j \ \forall i \neq j$

then A is <u>always</u> diagonalizable

Diagonalization and Left Eigenvectors (1.2)

$$T^{-1}AT = \Lambda$$

$$T^{-1}ATT^{-1} = \Lambda T^{-1}$$

$$T^{-1}A = \Lambda T^{-1}$$

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

$w_1^T \cdots w_n^T$ are <u>rows</u> of T^{-1}

$$w_i^T v_j = \delta_{ij} \quad \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$w_i^T A = \lambda_i w_i^T$$

rows of T^{-1} are the linearly independent <u>left</u> eigenvectors of A.

Diagonalization and Left Eigenvectors (2.2)

$$\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{T} \leftarrow \text{RIGHT EIGENVECTORS}$$

$$\underline{w_i^T v_j = \delta_{ij}}$$

$\uparrow$

BI-ORTHOGONALITY

$$\underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}}_{T^{-1}} \leftarrow \text{left eigenvectors}$$

$A = [a_1 \cdots a_n] \in \mathbb{R}^{n \times n}$ square, non-singular

$A^{-1} = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} \longrightarrow b_i^T a_j = \delta_{ij}$

$x = AA^{-1}x = [a_1 \cdots a_n] \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} x = (b_1^T x) a_1 + \cdots + (b_n^T x) a_n$

$x = A^{-1}Ax = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} [a_1 \cdots a_n] x = \begin{bmatrix} (a_1 x) b_1^T \\ \vdots \\ (a_n x) b_n^T \end{bmatrix}$

Modal Form (1.3)

Suppose that A is diagonalizable by T

define a new set of coordinates $x = T\tilde{x}$

$$\dot{x} = Ax \longrightarrow \dot{(T\tilde{x})} = AT\tilde{x}$$

$$\dot{\tilde{x}} = T^{-1}AT\hat{x} \Rightarrow \dot{\tilde{x}} = \underbrace{\Lambda}_{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}\tilde{x}$$

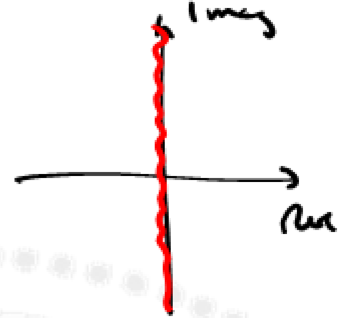
Modal Form (2.3)

n independent modes

$$\tilde{x}_i(t) = e^{\lambda_i t}\,\tilde{x}_i(0)$$

<u>MODAL FORM</u>

Modal Form (3.3)



$\dot{x} = Ax \qquad T^{-1}AT = \Lambda$

$x(t) = e^{At}x(0) \qquad \underline{e^{At} = Te^{\Lambda t}T^{-1}}$

$= Te^{\Lambda t}T^{-1}x(0)$

$= \sum_{i=1}^{n} e^{\lambda_i t}(w_i^T x(0))v_i$ ← reconstructs state from modal responses.

v_i ← modes or "mode shapes" of the system

w_i^T ← left eigenvectors decompose $x(0)$ into modal coordinates.

$e^{\lambda_i t}$ propagates the ith mode forward

Real Modal Form (1.3)

when eigenvalues of A are complex $\rightarrow$ T is also complex

$$S^{-1}AS = \mathrm{diag}\left(\Lambda_r, \begin{bmatrix}\sigma_{r+1} & \omega_{r+1}\\ -\omega_{r+1} & \sigma_{r+1}\end{bmatrix}, \dots, \begin{bmatrix}\sigma_n & \omega_n\\ -\omega_n & \sigma_n\end{bmatrix}\right)$$

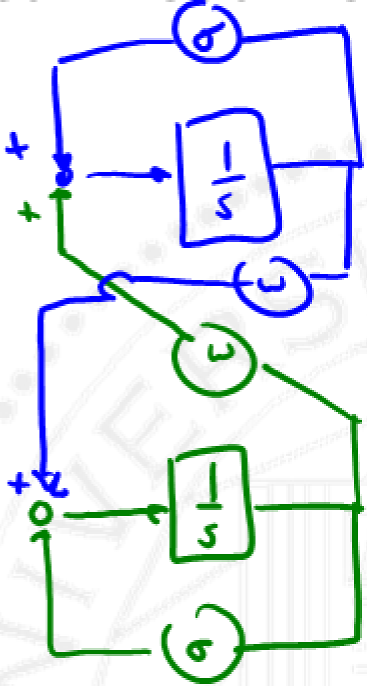
$$\det\left(sI - \begin{pmatrix}\sigma & \omega\\ -\omega & \sigma\end{pmatrix}\right) = 0 \rightarrow (s-\sigma)^2 + \omega^2$$

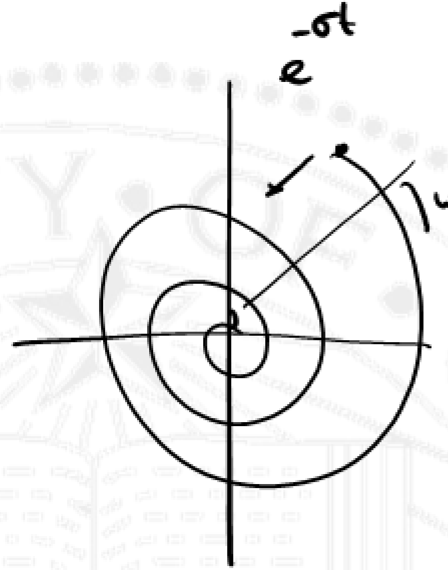
$$\underline{\lambda_i = \sigma \pm j\omega}$$

$$S^{-1}AS = \begin{bmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & \begin{bmatrix}\sigma & \omega\\ -\omega & \sigma\end{bmatrix} & \\ & & & & \ddots & \\ & & & & & [\ \ ]\end{bmatrix}$$

Real Modal Form (2.3)

$\dot{x} = \lambda x$





$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Real Modal Form (3.3)

$$T^{-1}AT = \Lambda \quad \therefore \quad A = T\Lambda T^{-1}$$

resolvent: $(sI - A)^{-1} = (s\underbrace{I}_{TT^{-1}} - T\Lambda T^{-1})^{-1}$

$$= (sTT^{-1} - T\Lambda T^{-1})^{-1}$$

$$= [T(sI - \Lambda)T^{-1}]^{-1}$$

$$= T(sI - \Lambda)^{-1}T^{-1}$$

$$[\lambda I - \Lambda]^{-1} = T(\lambda I - \Lambda)^{-1} T^{-1}$$

$$= T \begin{bmatrix} (\lambda - \lambda_1) & & & \\ & (\lambda - \lambda_2) & & 0 \\ & & \ddots & \\ 0 & & & (\lambda - \lambda_n) \end{bmatrix}^{-1} T^{-1}$$

$$= T \begin{bmatrix} \frac{1}{\lambda - \lambda_1} & & & \\ & \frac{1}{\lambda - \lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda - \lambda_n} \end{bmatrix} T^{-1}$$

$$= T \operatorname{diag}\left(\frac{1}{\lambda - \lambda_1}, \frac{1}{\lambda - \lambda_2}, \ldots \frac{1}{\lambda - \lambda_n} \right) T^{-1}$$

Solution via Diagonalization (1.3)

$$A^k = (T\Lambda T^{-1})^k = (T\Lambda T^{-1})(T\Lambda T^{-1})\cdots(T\Lambda T^{-1})$$

$$= T\Lambda T^{-1}T\Lambda T^{-1}T\Lambda \cdots T^{-1}T\Lambda T^{-1}$$

$$= T\Lambda^k T^{-1}$$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

if $k < 0$ only works if A is invertible

all $\lambda_i \neq 0$.

Solution via Diagonalization (2.3)

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \qquad A = T\Lambda T^{-1}, \quad I = TT^{-1}$$

$$= TT^{-1} + T\Lambda T^{-1} + \frac{T\Lambda T^{-1}T\Lambda T^{-1}}{2!} + \frac{T\Lambda T^{-1}T\Lambda T^{-1}T\Lambda T^{-1}}{3!} + \cdots$$

$$= TT^{-1} + T\Lambda T^{-1} + T\frac{\Lambda^2}{2!}T^{-1} + T\frac{\Lambda^3}{3!}T^{-1}$$

$$= T\left[I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!}\right]T^{-1} = T e^{\Lambda} T^{-1}$$

$$= T\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}T^{-1}$$

Solution via Diagonalization (3.3)

Any function which is analytic (power series)

$$f(A) = T \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} T^{-1}$$

"spectral mapping theorem"

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \cdots \beta_n A^n \qquad f: \mathbb{R} \to \mathbb{R}^n$$

$$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \cdots \beta_n a^n$$

Stability of Discrete-Time Systems (1.3)

for what $x(0) \longrightarrow x(t) = 0$ as $t \to \infty$

$\dot{x} = Ax \qquad x(0) = [0] \longrightarrow x(t) = 0 \ \forall t$.

$\text{Real}(\lambda_1) < 0 \ \ldots \ \text{Real}(\lambda_s) < 0$ ← stable eigvalues

$[v_1 \cdots v_s]$ ← stable eigvectors

$\text{Real}(\lambda_{s+1}) \geq 0 \ \ldots \ \text{Real}(\lambda_n) \geq 0$ ← unstable eigvalues

$[v_{s+1} \cdots v_n]$ ← unstable eigvectors .

$w_i^T x_0 = 0$

$\forall_{i = s+1 \ldots n}$

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} (w_i^T x(0)) v_i$$

$x(0) \in \text{span}\{v_1 \ldots v_s\}$ ← stable subspace

Stability of Discrete-Time Systems (2.3)

suppose A is diagonalizable — $x_{k+1} = A x_k$

$A = T \Lambda T^{-1}$ $\quad A^k = T \Lambda^k T^{-1}$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

$$x_k = A^k x(0) = \sum_{i=1}^{n} \lambda_i^k (w_i^T x(0)) v_i$$

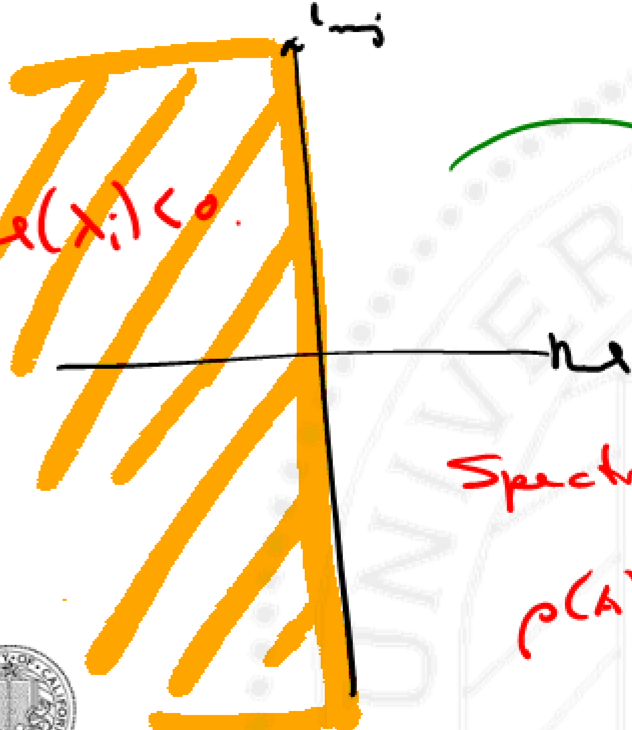
$\rightarrow x_k \to 0$ as $k \to \infty$

$$|\lambda_i| < 1 \ \forall i$$

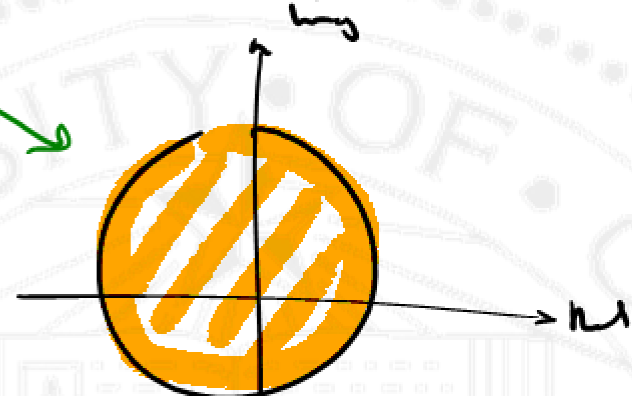
Stability of Discrete-Time Systems (3.3)

$\dot{x} = Av$

$x_{k+1} = A x_k$



$Re(\lambda_i) < 0$



Spectral Radius

$\rho(A) \doteq \max |\lambda_i|$

Questions?