$\dot{x} = Ax$ no inputs

Solution via Laplace Transform and Matrix Exponential

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Winter 2016

Complex Plane Visualization (2.2)

Suppose A has repeated eigenvalues such that

x_i can have repeated poles

$\lambda_1 \dots \lambda_r$ distinct $n_1 - n_r$ $\left(\sum n_i = n \right)$

$$x_i(t) = \sum_{j=1}^{r} p_{ij}(t)\, e^{\lambda_j t} \qquad \underline{t,\ t^2,\ t^3 \dots}$$

$p_{ij}(t)$ is a polynomial of degree $< n_j$

Stability (1.3)

$\dot{x} = Ax$ "stable" $e^{At} \to 0$ as $t \to \infty$

stable: $x(t) \to \{0\}$ as $t \to \infty$ no matter what $x(0)$.

all trajectories of $\dot{x} = Ax$ converge to $\{0\}$ as $t \to \infty$.

$\dot{x} = Ax$ "stable" iff all eigenvalues of A have negative real parts.

$$\boxed{\mathrm{Real}(\mathrm{Eig}(A)) = \mathrm{Real}(\lambda_i) < 0 \quad i = 1..n}$$

Stability (2.3)

IF

$\lim_{t \to \infty} p(t) e^{\lambda t} = 0$ for any polynomial $p(t)$

if $\mathrm{Real}(\lambda) < 0$.

Questions?

Eigenvectors and Diagonalization

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Eigenvectors and Diagonalization

- Eigenvectors
- Dynamic interpretation—invariant sets
- Complex eigenvectors and invariant planes
- Left eigenvectors
- Diagonalization
- Modal Form
- Discrete-time stability

Eigenvectors and Eigenvalues (1.3)

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

eigenvalues are points where the RESOLVANT is undefined

$v \in \mathbb{C}^n$ such that $(\lambda I - A)v = 0$

$$\underline{Av = \lambda v}$$

any such v is called a right eigenvector of A, corresponding to eigenvalue λ.

Eigenvectors and Eigenvalues (2.3)

$w \in \mathbb{C}^n$ such that $w^T(\lambda I - A) = 0$

$$w^T A = \lambda w^T$$

w defined as a left eigenvector of A, corresponding to frequency λ.

Note: $A \in \mathbb{R}^{n \times n}$ $\lambda, v, w \in \mathbb{C}$

$v \in \mathbb{C}^n$ $\lambda \in \mathbb{C}$

$\downarrow$

$\bar{v} \in \mathbb{C}^n$ $\bar{\lambda} \in \mathbb{C}$

$\overline{[Av = \lambda v]}$

$\bar{A}\bar{v} = \bar{\lambda}\bar{v}$

$\downarrow$

$A\bar{v} = \bar{\lambda}\bar{v}$

$\bar{v}, v^*, v^H$

MATLAB v'

Eigenvectors and Eigenvalues (3.3)

$$Av = \lambda v$$

$$\dot{x} = Ax \qquad x_0 = v$$

$$\dot{x} = Ax = Av$$

$$\dot{x} = \lambda v$$

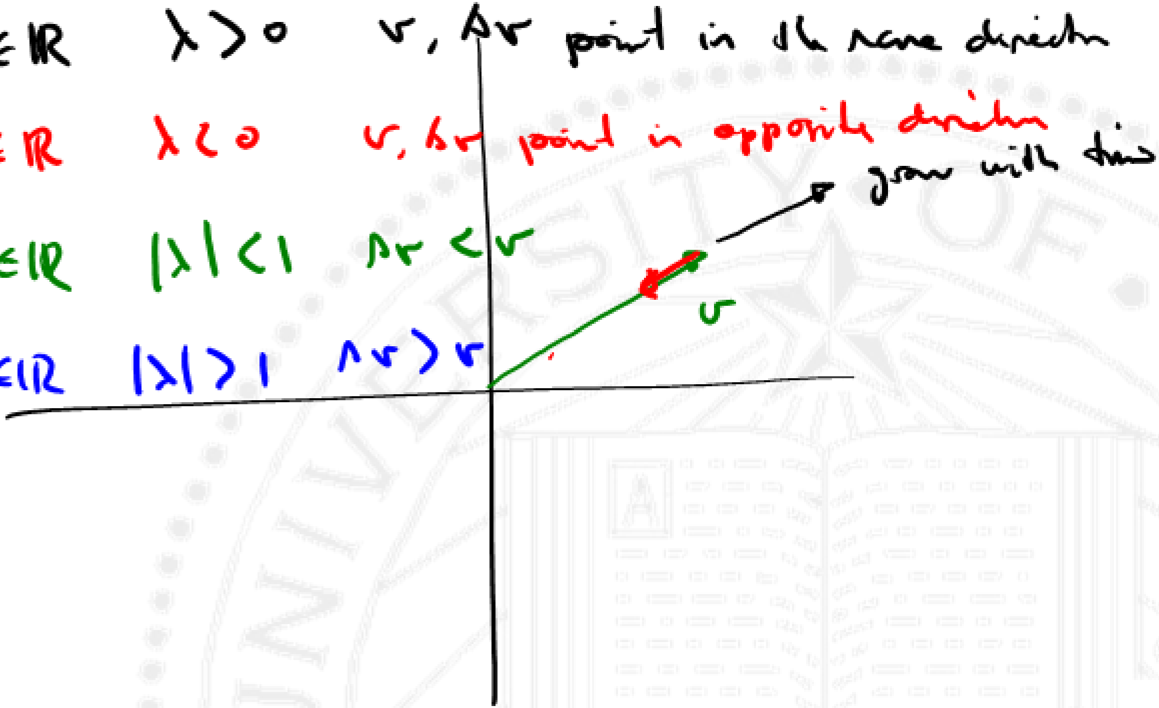
Scaling Interpretation (1.2)

$\lambda \in \mathbb{R}$ $\lambda > 0$ v, Av point in the same direction

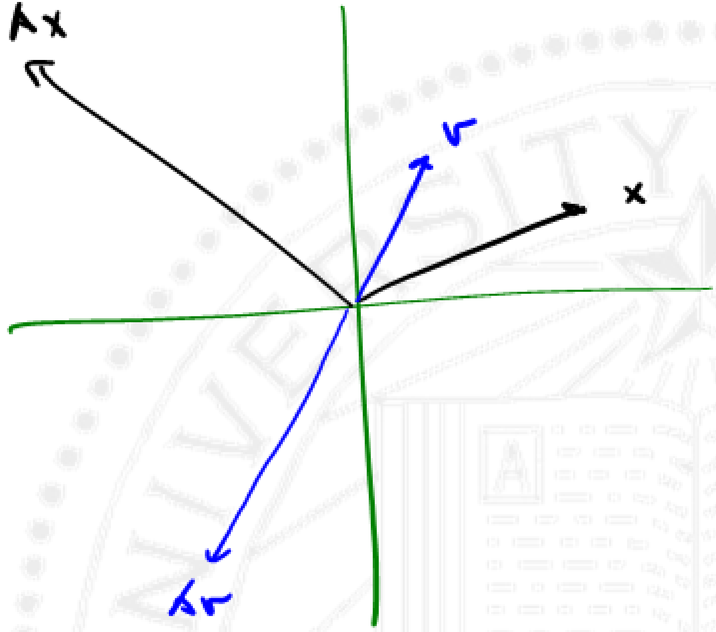
$\lambda \in \mathbb{R}$ $\lambda < 0$ v, Av point in opposite direction

$\lambda \in \mathbb{R}$ $|\lambda| < 1$ $Av < v$

$\lambda \in \mathbb{R}$ $|\lambda| > 1$ $Av > v$



Scaling Interpretation (2.2)



$x(t) = e^{\lambda t} v$ <u>mode</u> of the system.

Dynamic Interpretation (1.2)

$Av = \lambda v \qquad v \neq 0.$

$\dot{x} = Ax \qquad x(0) = v \qquad \underline{x(t) = e^{\lambda t} v}$

$$x(t) = e^{At} v = \left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right] v$$

$$= v + \underbrace{Av}_{\lambda v}t + A\underbrace{(Av)}_{\lambda v}\frac{t^2}{2!} + \underbrace{A(A)Av}_{\lambda^3 v}\frac{t^3}{3!}$$

$$\lambda v t \qquad \lambda^2 v$$

$$= v + \lambda t v + \frac{\lambda^2t^2}{2!}v + \frac{\lambda^3t^3}{3!}v$$

$$= \left[1 + \lambda t + \frac{\lambda^2t^2}{2!} + \frac{\lambda^3t^3}{3!} \dots\right] v = e^{\lambda t} v$$

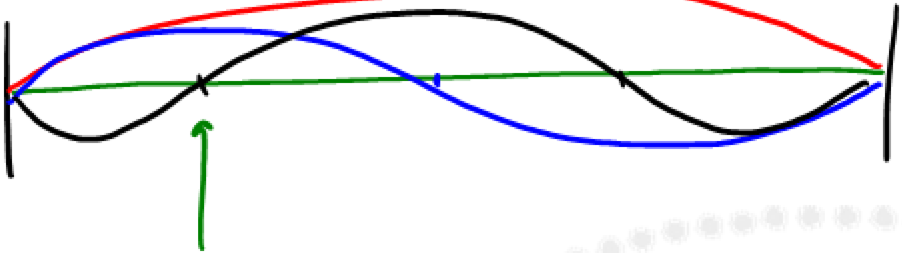
Dynamic Interpretation (2.2)

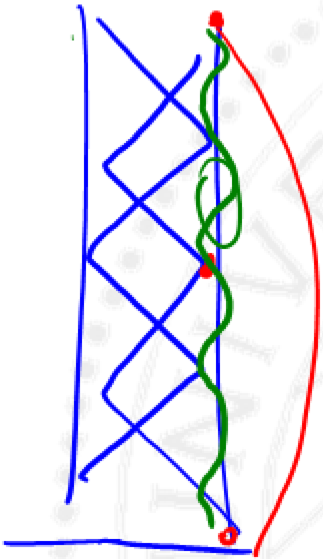
$x(t) = e^{\lambda t} v$ ← mode system.

if the initial state, $x(0) = v$ resulting motion

is always on line spanned by v

$x(t) = e^{\lambda t} v$ is mode associated with λ_i



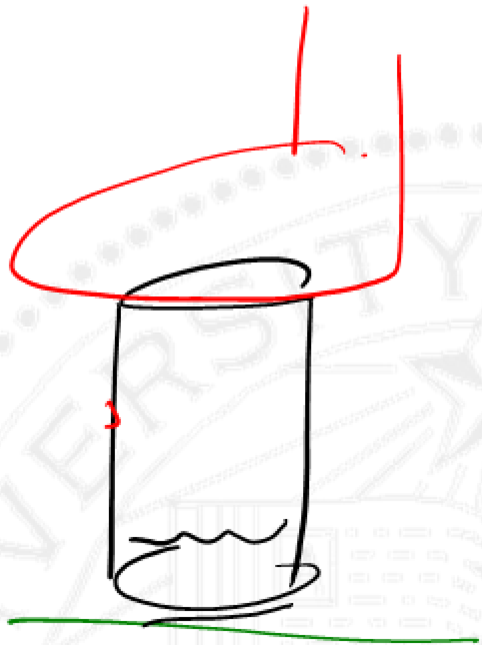


$\omega_0 \downarrow$

$\lambda \uparrow$

min mass $\{v_i\}_j$

max λ_j

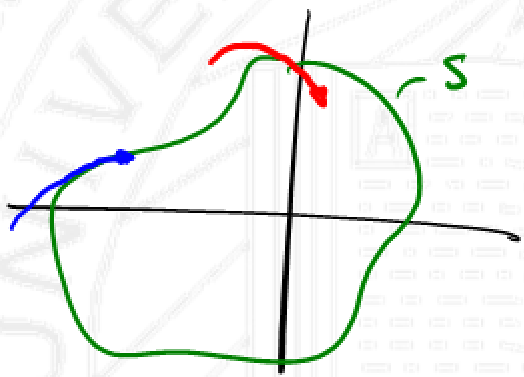


Invariant Sets (1.3)

a set $S \subseteq \mathbb{R}^n$ <u>invariant</u> under $\dot{x} = Ax$

if whenever $x(t) \in S$ then $x(\tau) \in S \;\; \forall \; \tau > t$.

once a trajectory enters S it remains in S.



Invariant Sets (2.3)

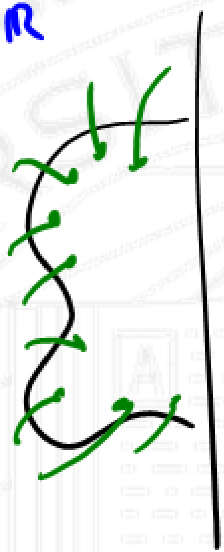
Vector Field Interpretation

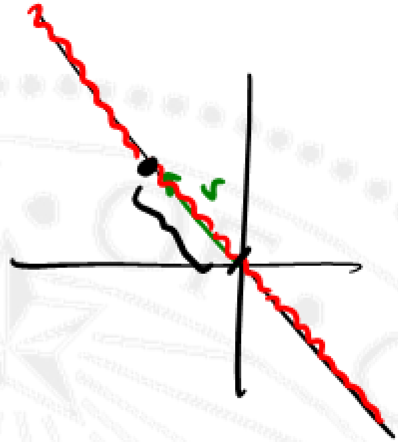
$Av = \lambda v \quad v \neq 0 \quad \lambda \in \mathbb{R}$

line $\{tv \mid t \in \mathbb{R}\}$ is invariant.

$\lambda < 0$ line segment

$\{tv \mid 0 \leq t \leq a\}$ invariant

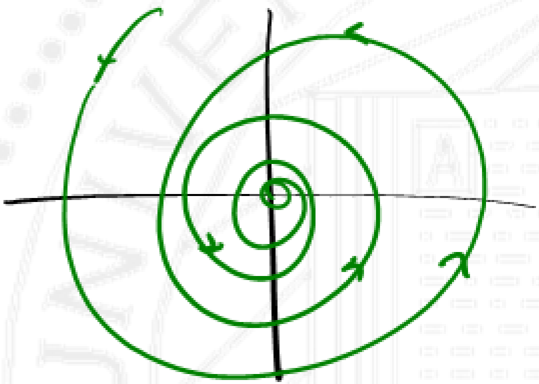




Invariant Sets (3.3)

$\dot{x} = \lambda x$ $\qquad \lambda = -0.1 \pm 3j$

$$e^{0.1t} \cos(3t + \phi)$$



Complex Eigenvectors (1.2)

$Av = \lambda v \quad v \neq 0 \quad \lambda$ complex

for $a \in \mathbb{C}$ (complex) trajectory $\underbrace{a e^{\lambda t} v}_{\text{complex}}$ satisfies $\dot{x} = Ax$

oscillating behavior

$x(t) = \mathrm{Real}(a e^{\lambda t} v) \qquad e^{\lambda t} = e^{\sigma t}(\cos \omega t + j \sin \omega t)$

$$= e^{\sigma t} \underbrace{\begin{bmatrix} v_{real} & v_{imag} \end{bmatrix}}_{\substack{\text{describe a plane} \\ \text{in } \mathbb{R}^n}} \underbrace{\begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \leftarrow \text{constants}$$

Complex Eigenvectors (2.2)

$v = v_{real} + j v_{imag}$

$\lambda = \sigma + j\omega$

$a = \alpha - j\beta$

trajectory stays in an <u>invariant</u> plane

span$\{v_{real}, v_{imag}\}$

σ - gives the logarithmic growth/decay

ω - gives the angular velocity of rotation in the plane.

Dynamic Intepretation: Left Eigenvectors (1.2)

$w^T A = \lambda w^T \qquad w \neq 0 \qquad w^T x(t) =$ scalar fn of time

$\dot{x} = Ax$

$$\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x$$

$$= \lambda w^T x = \underline{\lambda (w^T x)}$$

$w^T x$ satisfies a D.E. $\frac{d}{dt}(w^T x) = \lambda (w^T x)$

$$w^T x(t) = e^{\lambda t} w^T x(0)$$

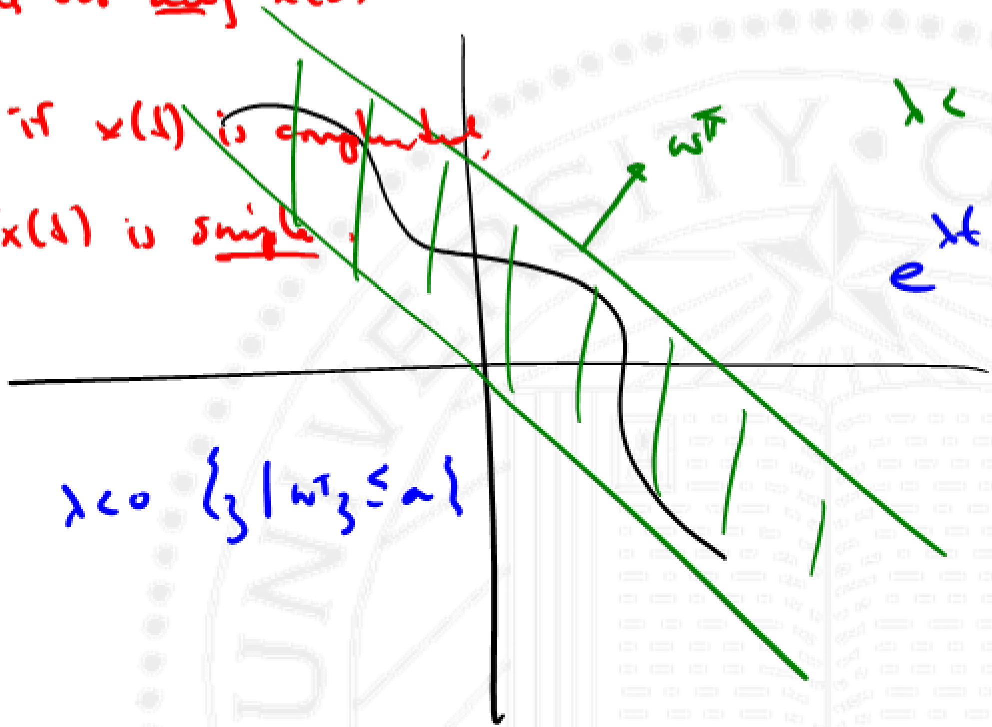
$\dot{x} = \lambda x \qquad x(0) = x_0 .$

Dynamic Intepretation: Left Eigenvectors (2.2)

holds for any $x(0)$

even if $x(t)$ is complicated,

$w^T x(t)$ is simple.



$\lambda < 0$

$e^{\lambda t}$

$\lambda \in \mathbb{R}$ $\lambda < 0$ $\{z \mid w^T z \leq a\}$

Eigenvector Summary

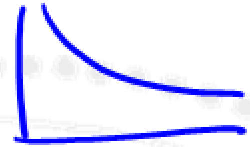
- Right eigenvectors are initial conditions such that resulting motion is very simple

- Left eigenvectors give LINEAR FUNCTIONS of the state that are simple for all $x(0)$.

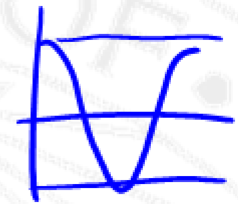
Eigenvalue/vector example (1.4)

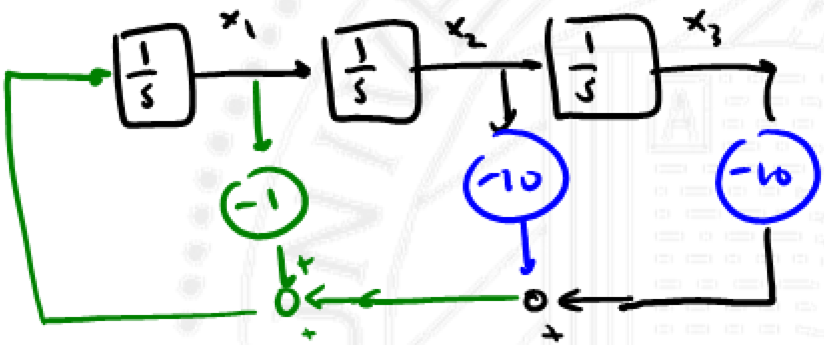
$$\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

e^{-t}



$\cos(\sqrt{10}\,t)$





$\lambda: -1, \pm j\sqrt{10}$

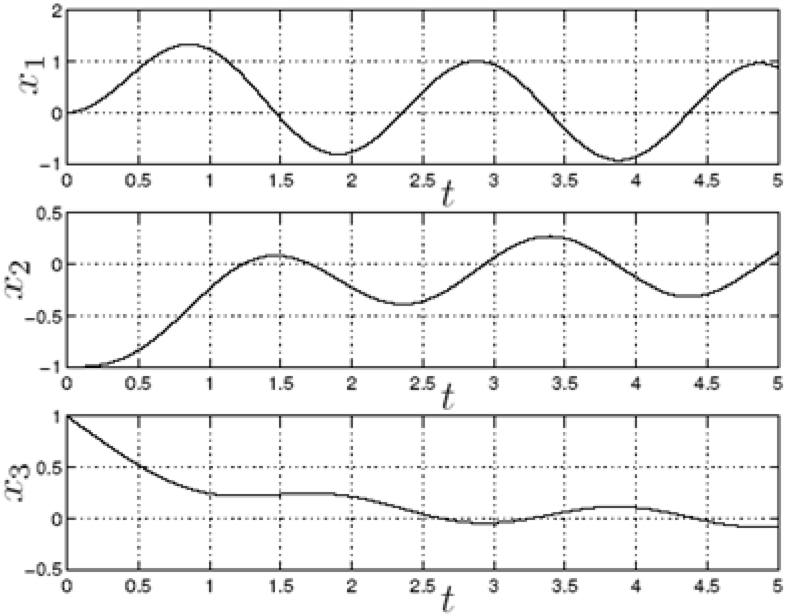
$$\chi(s) = \lambda^3 + \lambda^2 + 10\lambda + 10 = (\lambda+1)(\lambda^2+10)$$

Eigenvalue/vector example (2.4)

$x(0) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ e^{-t}

$\cos(\sqrt{6}t)$

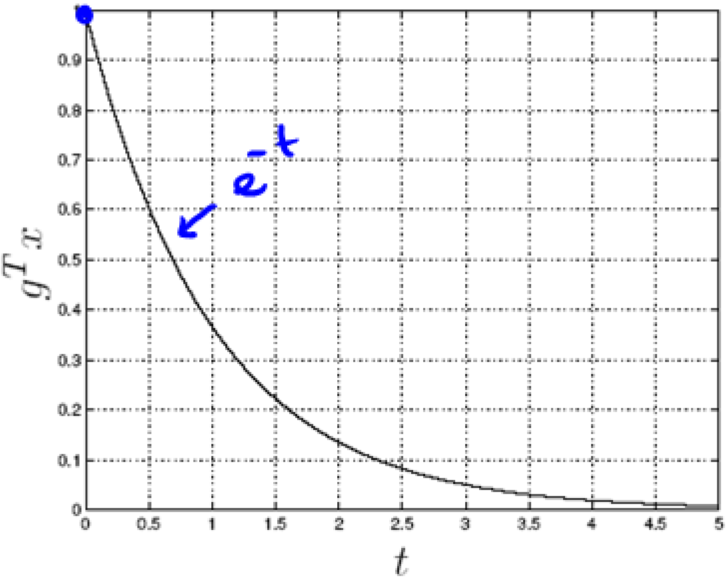
$\lambda = -1$ $w = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}$



Eigenvalue/vector example (3.4)

$[0.1 \quad 0 \quad 1]\, x(t)$

$x_0 = 0.1\, x(0) + 1\, x_3(0)$



Eigenvalue/vector example (4.4)

$\lambda: j\sqrt{16}$

$$v = \begin{bmatrix} -0.559 + j\,0.771 \\ 0.244 + j0.175 \\ 0.055 - j\,0.037 \end{bmatrix}$$

$$v_{re} = \begin{bmatrix} -0.559 \\ 0.244 \\ 0.055 \end{bmatrix} \qquad v_{im} = \begin{bmatrix} 0.771 \\ 0.175 \\ 0.037 \end{bmatrix}$$

$x(0) = v_{real}$