$\dot{x} = Ax$ no inputs

Solution via Laplace Transform and Matrix Exponential

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5 PM

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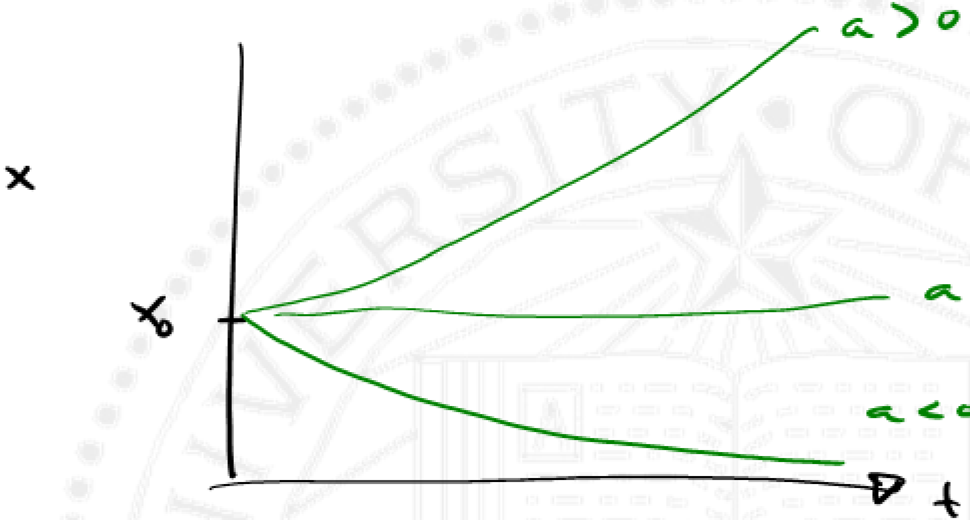
$\dot{x} = ax$ ← scalar $\quad x(t) = x_0 e^{at}$



$\dot{\underline{x}} = A\underline{x} \longrightarrow \underline{x}(t) = e^{At} x_0$

Solution via Laplace Transform and Matrix Exponential

- Laplace Tranform
- Solving x’ = Ax via Laplace Tranform
- State transition matrix
- Matrix Exponential
- Qualitative behavior and Stability

Laplace Transform of a Matrix (1.3)

$$\mathcal{L}\{x(t)\} \triangleq \int_0^\infty x(t)e^{-st}\,dt \triangleq X(s) \qquad x: \mathbb{R}_+ \to \mathbb{R}$$

$$X: \mathbb{C} \to \mathbb{C}$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$$

$$\mathcal{L}\{\dot{x} = ax\} = sX(s) - x_0 = aX(s) \to (s-a)X(s) = x_0$$

$$X(s) = \frac{x_0}{s-a}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0}{s-a}\right\} = \underline{x_0 e^{at}}$$

Laplace Transform of a Matrix (2.3)

$g : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$ ← trajectory, matrix valued signal

$Z = \mathcal{L}(g)$ where $Z : D \subseteq \mathbb{C} \to \mathbb{C}^{p\times q}$

complex

$$Z(s) = \int_0^\infty e^{-st} g(t)\,dt$$

$$Z(s) = \int_0^\infty g(t) e^{-st}\,dt$$

$D \triangleq$ domain of convergence

D includes at least $\{ s \mid \mathrm{Re}(s) > a \}$

where a: $|g_{ij}(t)| \le \alpha e^{at}$ $\quad t > 0 \quad i = 1..p \quad j = 1..q$

Laplace Transform of a Matrix (3.3)

$$\mathcal{L}\{g(t)\} = Z(s)$$

$$\mathbb{C}^{p \times q}$$

$$\mathbb{R}_+ \rightarrow \mathbb{R}^{p \times q}$$

lower case (t) ← time   uppercase (s) ← freq.

Derivative Property (1.2)

$$\mathcal{L}\{\dot{y}\} = sY(s) - y(0) \qquad \int u\,dv = uv - \int v\,du$$

$$\mathcal{L}\{\dot{y}\} = \int_0^\infty \underbrace{\dot{y}(t)}_{dv}\,\underbrace{e^{-st}}_{u}\,dt = e^{-st}y(t)\Big|_{t=0}^{\infty} + \int_0^\infty y(t)e^{-st}(-s\,dt)$$

$$= 0 - 1\,y(0) + s\underbrace{\int_0^\infty y(t)e^{-st}\,dt}_{Y(s)}$$

$$\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$$

$$\mathcal{L}\{\ddot{y}\} = s^2Y(s) - s\,y(0) - \dot{y}(0).$$

Laplace transform solution of $\dot{x}$=Ax (1.3)

$$\mathcal{L}\{\dot{x}=Ax\} = sX(s)-x(0)=AX(s)$$

$$sX(s)-AX(s)=x(0)$$

$$[sI-A]X(s)=x(0)$$

$$X(s)=[sI-A]^{-1}x(0)$$

Laplace transform solution of $\dot{x} = Ax$(1.3)

$\dot{x} = Ax$

$\mathcal{L}\{Ax\} = A\mathcal{L}\{x\}$ $\quad A \in \mathbb{R}^{n\times n}$

$sX(s) - x(0) = \overbrace{AX(s)}$ complex n×n function

$(sI - A)X(s) = x(0)$

$X(s) = \underbrace{[sI - A]^{-1}}_{\text{Resolvent of } A}x_0$ $\sim$ 1850

$s \in \mathbb{C}$ except for eigenvalues of A.

Laplace transform solution of $\dot{x} = Ax$ (2.3)

$[sI - A]^{-1} \leftarrow$ resolvant

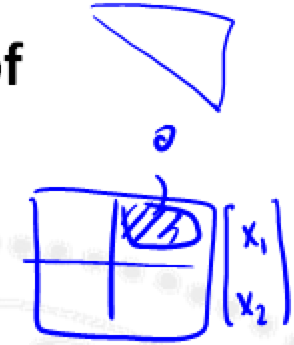
defined $s \in \mathbb{C}$ except for eigenvalues of A.

$\det(sI - A) = 0$

$X(s) = [sI - A]^{-1} x_0$

$x(t) = \mathcal{L}^{-1}\{[sI - A]^{-1} x_0\}$

Laplace transform solution of $\dot{x} = Ax$ (3.3)



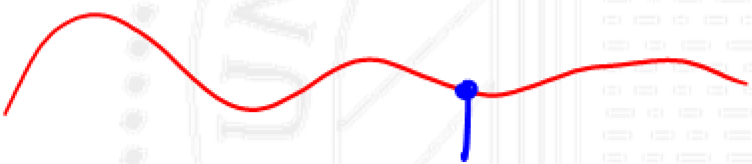
$$\mathcal{L}^{-1}\{X(s) = [sI-A]^{-1}x_0\}$$

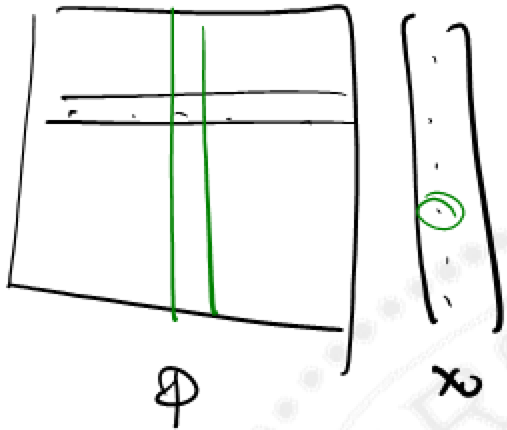
$$x(t) = \underbrace{\mathcal{L}^{-1}\{[sI-A]^{-1}\}}_{\Phi(t)\ \sim\ \mathbb{R}^{n\times n}} x_0$$

$$\boxed{x(t) = \Phi(t)\, x_0}$$

$$\Phi(t) \triangleq \mathcal{L}^{-1}\{[sI-A]^{-1}\}$$

state transition matrix



$x(t) =$ 

$\phi(t) \to 0$ as $t \to \infty$

STABILITY

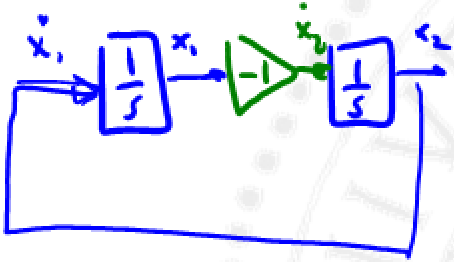
Autonomous LDS example (1.3)
Harmonic Oscillator

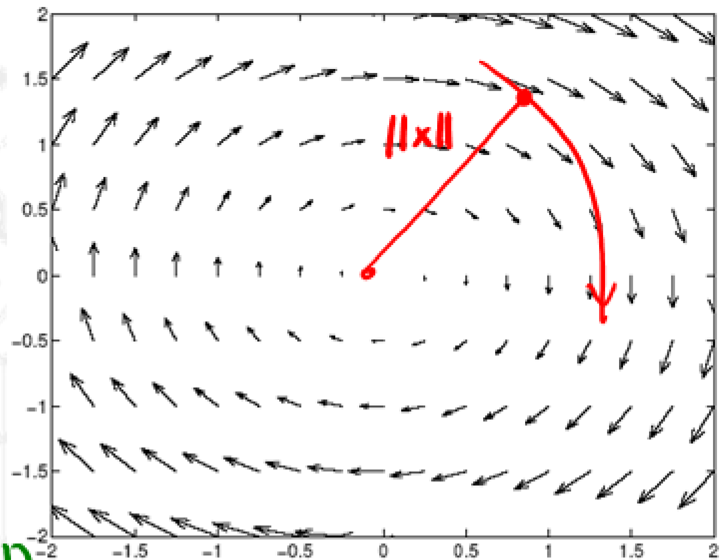
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

$\lambda = \pm j$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1$





$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \longrightarrow \mathcal{L}^{-1}\{\} \rightarrow \Phi = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Autonomous LDS example (2.3)
Harmonic Oscillator

$$\dot{x} = ax$$

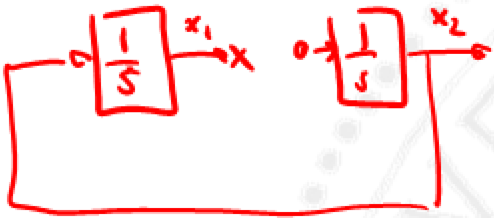
$$\dot{x} = Ax \longrightarrow x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x_0$$

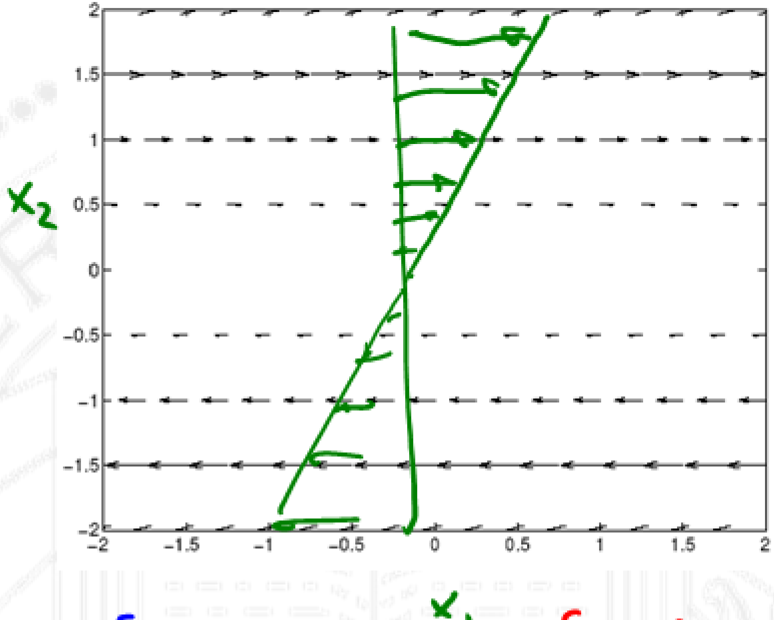
Autonomous LDS example (1.3)
Double Integrator

$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$

$\dot{x}_2 = 0$

$\dot{x}_1 = x_2$





$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} \rightarrow (sI - A)^{-1} = \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \rightarrow \Phi = \mathcal{L}^{-1}\{\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

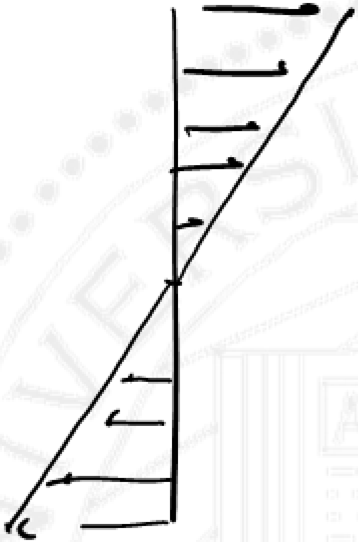
Autonomous LDS example (2.3)
Double Integrator

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x_0$$

$\mathrm{eig}(sI - A) = (0, 0)$

$\det(sI - A) = 0$ when $s = 0$. $\frac{1}{s} \cdot \frac{1}{s}$

Autonomous LDS example (3.3)
Double Integrator



Characteristic Polynomial (1.2)

$\chi(s) \triangleq \det(sI - A)$ characteristic polynomial of A $[\Delta(s)]$

$\chi(s)$ is a polynomial of degree n

(monic polynomial)

$s^n + (\Sigma a_{ii}) s^{n-1} + \ldots + (\Pi a_{ii})$

$\det \begin{bmatrix} (s - a_{11}) & a_{12} & \cdots \\ a_{21} & (s - a_{22}) & \\ & & \ddots \end{bmatrix}$

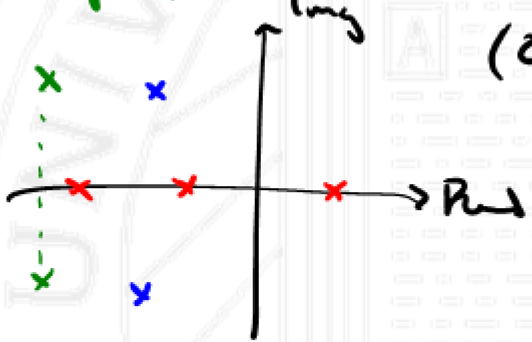
n eigenvalues (if we count multiplicity of roots)

Characteristic Polynomial (2.2)

roots of $\chi(s)$ are the eigenvalues of A.

$\chi(s)$ has only <u>REAL</u> coefficients, eigenvalues are <u>either</u> real, or they come in complex conjugate pairs.

$(\sigma \pm j\omega)$



i,j entry of the resolvent $(sI-A)^{-1}$

Cramer's Rule for matrices $\dfrac{\mathrm{adj}(sI-A)}{\det(sI-A)}$

$$(-1)^{i+j}\frac{\det \Delta_{ij}}{\det(sI-A)}$$

$\leftarrow$ degree $< n$ $(n-1)$

$\leftarrow \mathcal{X}(s) \leftarrow$ degree n

$\Delta_{ij} = sI - A$ with j^{th} row & i^{th} column deleted

rational functions in "s".

The Matrix Exponential (1.3)

$$\frac{1}{1-c} = 1 + c + c^2 + c^3 + \ldots$$ works if $|c| < 1$.

$$[I - C]^{-1} = I + C + C^2 + C^3 + \ldots$$ if series converges.

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{1}{s}\left[I + \frac{A}{s} + \frac{A^2}{s^2} + \frac{A^3}{s^3} + \ldots\right]$$

$$= \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \frac{A^3}{s^4} + \ldots$$ converge as long as $|s|$ is big enough

$$\Phi = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \mathcal{L}^{-1}\left\{\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots\right\}$$

$1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} \cdots = e^{at}$

The Matrix Exponential (2.3)

$$\Phi = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \mathcal{L}^{-1}\left\{\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{I}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{A^2}{s^3}\right\} + \cdots$$

$$= I\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + A\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + A^2\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + \cdots$$

$$= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$M \in \mathbb{R}^{n\times n}$

MATRIX EXPONENTIAL: $e^M \triangleq I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$

The Matrix Exponential (3.3)

$\Phi(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = e^{At}$

$\boxed{x(t) = e^{At} x_0}$ ← non-trivial overloading

$\exp(A) = \begin{bmatrix} e^{a_{11}} & e^{a_{12}} & \cdots \\ \vdots & & \end{bmatrix}$ elmnt by elmnt

$\underline{\operatorname{expm}(A)}$ ← matrix exponential

Matrix Exponential Solution of A-LDS (1.5)

$e^a e^b = e^{(a+b)}$ ← scalar world $ab = ba$

$ab = 0$

$e^A e^B \neq e^{(A+B)}$ ← matrix $AB = \phi$.

$AB \neq BA$.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$e^A = \begin{bmatrix} 0.59 & 0.84 \\ -0.84 & 0.57 \end{bmatrix} \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Matrix Exponential Solution of A-LDS (2.5)

$$e^{A} e^{B} = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

$$\neq$$

$$e^{(A+B)} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix}$$

$$\boxed{e^{A} e^{B} = e^{(A+B)} \quad \text{if} \quad AB = BA}$$

Matrix Exponential Solution of A-LDS (3.5)

$t, s \in \mathbb{R}$

$$e^{At} \cdot e^{As} = e^{A(s+t)}$$

$At \cdot As = stA^2$

$As \cdot At = stA^2$

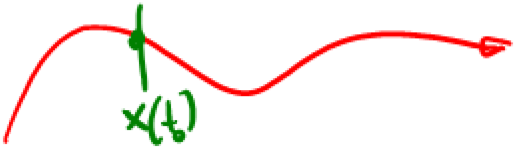
$t = 1 \quad s = -1$

$$e^{A(-1)} e^{A(1)} = e^{0} = I.$$

$$\boxed{\left(e^{At}\right)^{-1} = e^{-At}}$$

e^{At} non-singular.

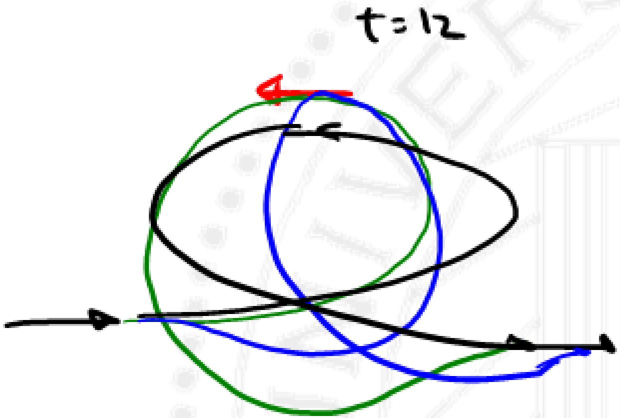
Matrix Exponential Solution of A-LDS (4.5)



$\dot{x} = Ax$

$x(50)$ ______

$x_0 = e^{-A(50)} x_{50}$



Matrix Exponential Solution of A-LDS (5.5)

$$e^{At} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (sI - A)^{-1} : e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$t = 2$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \underline{I + At} + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Time Transfer Property (1.2)

$\dot{x} = Bx \quad 0 \le t < 1$

$x(0) = x_0$.

$\dot{x} = Ax \quad 1 \le t < \infty$

$x(t) = e^{A(t-1)} x(1)$

$x(1) = e^{B(1)} x_0$

$$x(t) = \begin{cases} e^{Bt} x_0 & t < 1 \\ e^{A(t-1)} e^{B} x_0 & t \ge 1 \end{cases}$$

$$\underline{\underline{x(t+\tau) = e^{At} x(\tau)}}$$

Time Transfer Property (2.2)

$$\dot{x} = Ax \longrightarrow x(t) = \Phi(t)\, x_0 = e^{At} x_0$$

e^{At} propagates initial state into state at time t.

$$x(t+\tau) = e^{At} x(\tau) \qquad z(t) = x(t+\tau)$$

Forwards or Backwards in time.

$$\dot{x} = Ax \qquad x(t+\tau) = x(\tau) + t\dot{x}(\tau) = (I + At)x(\tau)$$

Forward integrate

$$x(t+\tau) = e^{At}x(\tau) = \left[I + At + \frac{A^2t^2}{2!} + \dots \right] x(\tau)$$

$$x(t+\tau) = \left[I - At \right]^{-1} x(\tau)$$

$$I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{4} + \dots$$

Sampling a continuous-time system (1.2)

$\dot{x} = Ax$

Sample x @ $t_1 \le t_2 \le t_3 \cdots$ $\qquad z_k \doteq x(t_k)$

$$z_{k+1} = e^{A(t_{k+1} - t_k)} z_k$$

$-\!\!\times\!\!-$ h

$$t_{k+1} - t_k = h \quad \forall k$$

$$\boxed{z_{k+1} = e^{Ah} z_k}$$

Discrete Version of CDS

Piecewise Constant System (1.2)

$$A(t) : \begin{cases} A_0 & 0 < t < t_1 \\ A_1 & t_1 < t < t_2 \\ A_2 & \\ \vdots & \end{cases}$$

jump linear system

power system (multi variables)

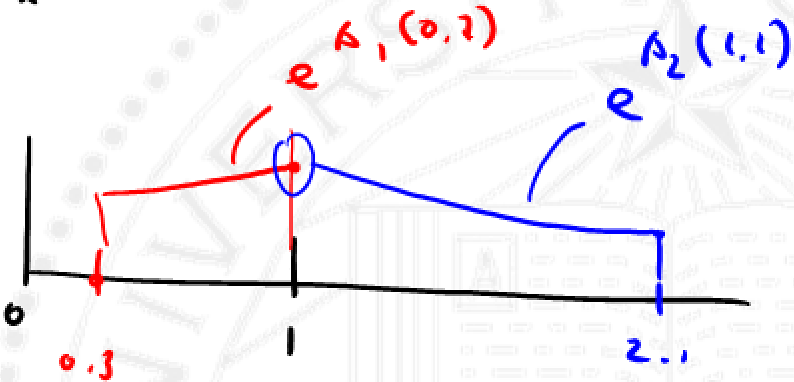
hybrid systems

$$x(t) = e^{A_i(t-t_i)} e^{A(t_i - t_{i-1})} \cdots e^{A(t_3-t_2)} e^{A(t_2-t_1)} e^{At_1} x_0$$

Piecewise Constant System (2.2)

$\dot{x} = A_1 x \qquad 0 < t < 1$

$\dot{x} = A_2 x \qquad t > 1$



Qualitative behavior of *x(t)* (1.3)

$\dot{x} = Ax \qquad x(t) \in \mathbb{R}^n$

$$x(t) = \underbrace{e^{At}}_{\text{Time}} x_0 \qquad X(s) = \underbrace{[sI - A]^{-1}}_{\text{Freq.}} x_0$$

i^{th} component of $X_i(s) = \dfrac{a_i(s)}{\mathcal{X}(s)}$ $\leftarrow \det(sI - A)$

a_i polynomial in s, degree $< n$

Poles of X_i are the eigenvalues of A.

Qualitative behavior of $x(t)$ (2.3)

Assume we have distinct eigenvalues $\lambda_i \neq \lambda_j$

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

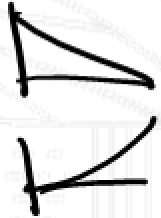
β_{ij} depends linearly on $x(0)$

$\dot{x} = Ax$ has a response that is a sum of the (complex) frequencies of A.

Qualitative behavior of $x(t)$ (3.3)

Eigenvalues of A give the exponents that CAN occur in exponentials.

Real Eigenvalues: λ corresponds to exponential growth or decay



Complex eigenvalues $\lambda = \sigma \pm j\omega$ corresponds to a decaying or growing sinusoidal term

$$e^{\sigma t} \cos(\omega t + \phi)$$

Real(λ) exponential envelope Im(λ) freq. of oscillation

$\mathcal{X}(s) = \det(sI - A) = 0$

Complex Plane Visualization (1.2)