Autonomous Linear Dynamical Systems

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Winter 2016

MIDTERM ~ OUT AFTER CLASS ON THURSDAY

COME BACK ON FRIDAY @ 2PM.

RULES OF MIDTERM — OPEN BOOK, NOTES, MATLAB

WEBSITES, WIKIPEDIA...

EXCEPT NO HUMAN/AI HELP

NO PUBLISHED SOLUTIONS

Autonomous Linear Dynamical Systems

$\dot{x} = Ax + B\overset{0}{\cancel{u}}$

- Autonomous Linear Dynamics Systems
- Examples
- Higher order systems
- Linearization near equilibrium point
- Linearization along trajectory

Autonomous Linear Dynamical Systems (1.4)

continuous time autonomous linear dynamical system

$$\dot{x} = Ax + B\cancel{u}$$

0. autonomous

C.T.I. $A \neq f(t)$

$x : \mathbb{R} \longrightarrow \mathbb{R}^n$

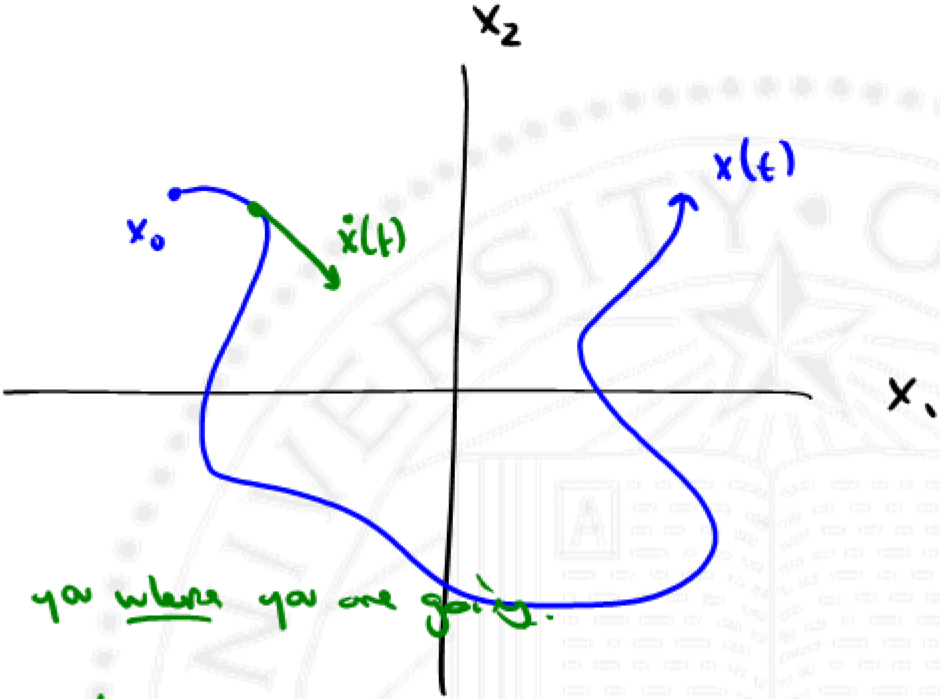
$x(t) \in \mathbb{R}^n$ called "state"

x_i called i^{th} state

n – state dimension, <u>order</u>, number of states.

$A(t) \triangleq$ dynamics matrix $\in \mathbb{R}^{n \times n}$

Autonomous Linear Dynamical Systems (2.4)



A tells you where you are going.

$$\dot{x} = \frac{dx}{dt}$$

$x(t)$ trajectory

$\dot{x}(t)$ tangent to trajectory

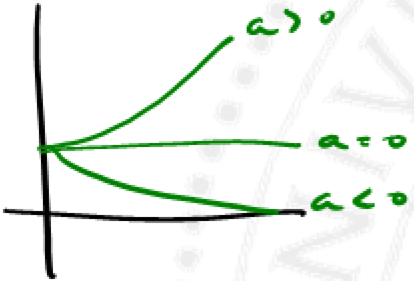
Autonomous Linear Dynamical Systems (3.4)

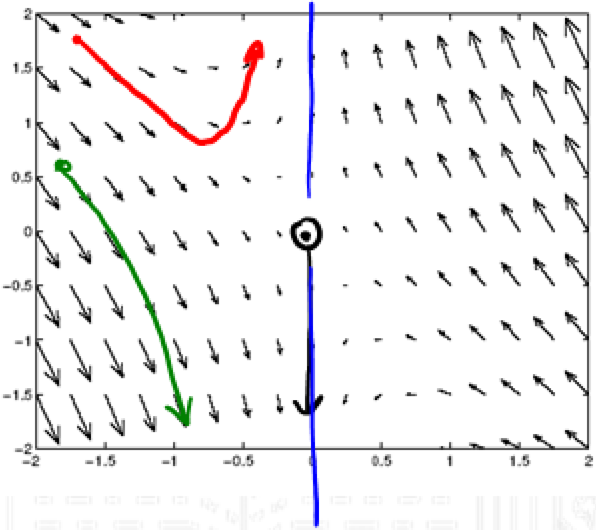
$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x$$

VECTOR FIELD

PHASE PLANE / $\mathbb{R}^2$

NON-OSCILLATORY BEHAVIOR

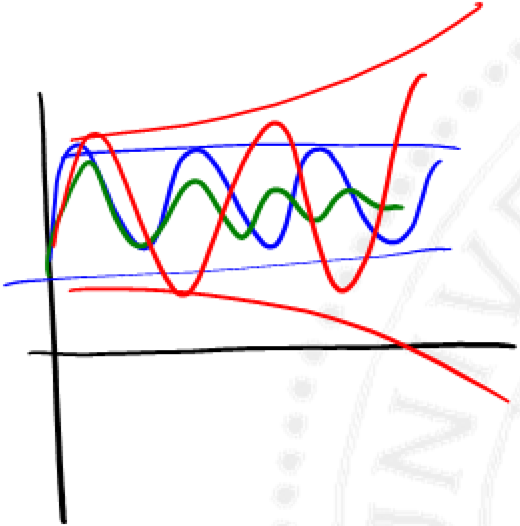


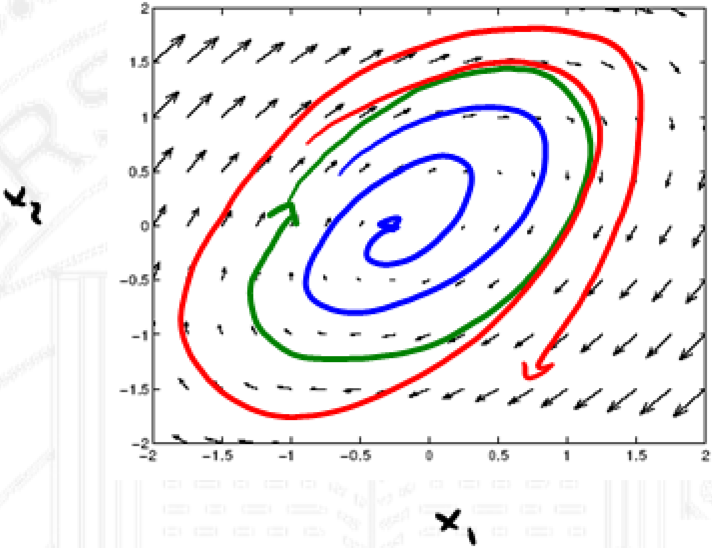


$$\dot{x} = ax \longrightarrow x(t) = x_0 e^{at}$$

Autonomous Linear Dynamical Systems (4.4)

$$\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x$$

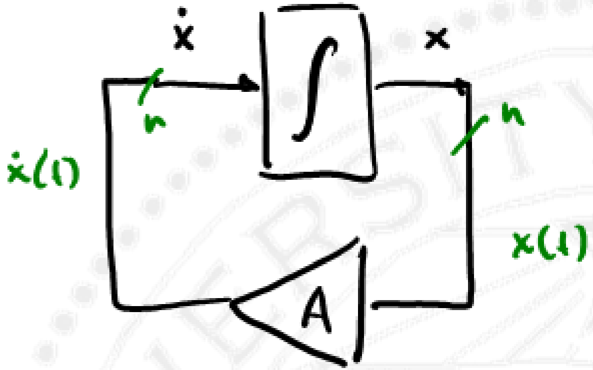




Autonomous LDS Block Diagram (1.3)

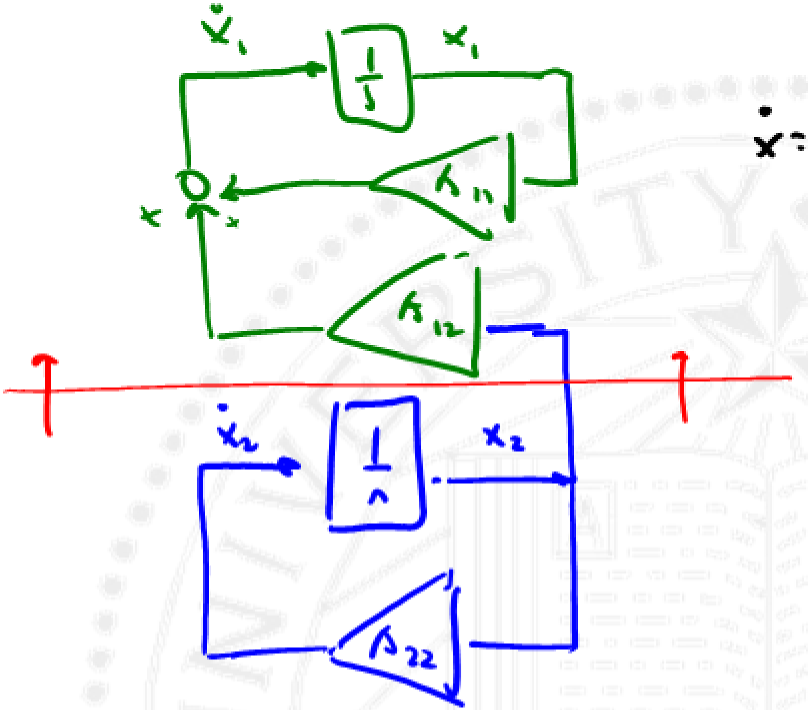
$\dot{x} = Ax$

$$\boxed{\frac{1}{\lambda}} \rightarrow \begin{bmatrix} \frac{1}{\lambda} & & \\ & \ddots & \\ & & \frac{1}{\lambda} \end{bmatrix}$$



coupling comes from Dynamics Matrix "A".

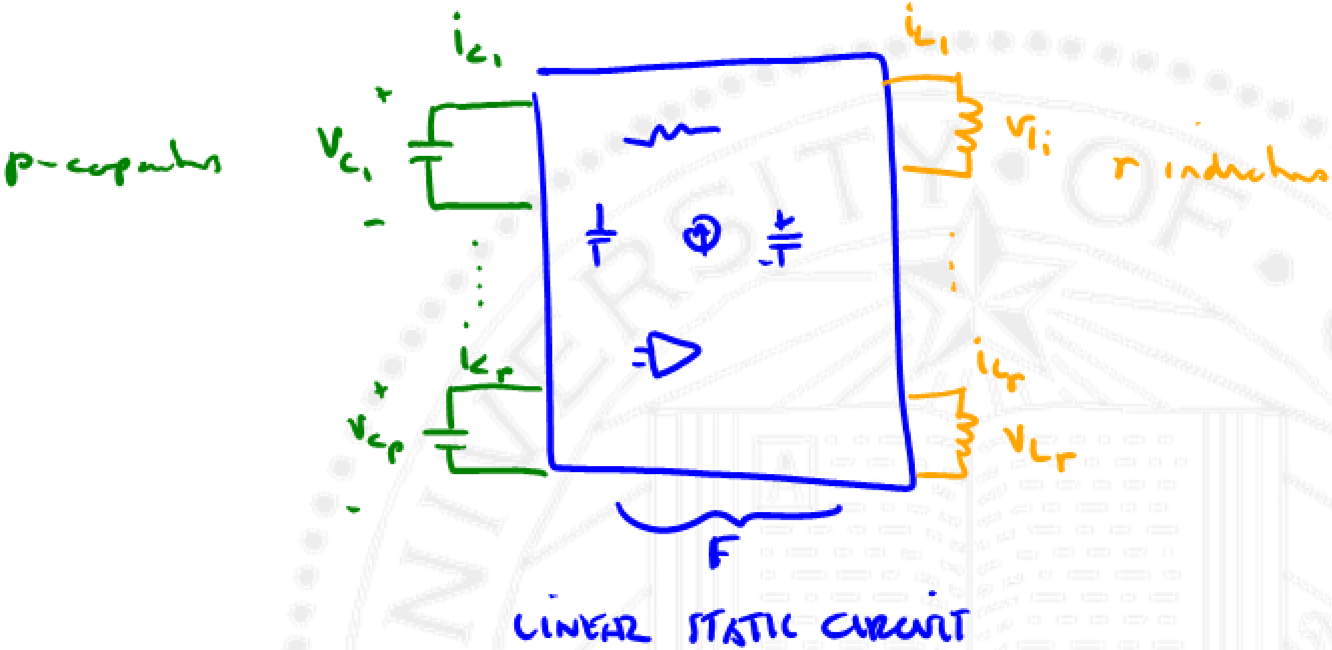
Autonomous LDS Block Diagram (2.3)



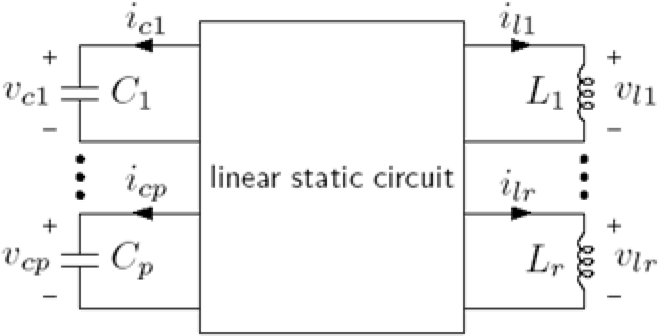
$$\dot{x} = \left[\begin{array}{c|c} \lambda_{11} & \lambda_{12} \\ \hline 0 & \lambda_{22} \end{array}\right] x$$

$$\boxed{\frac{1}{s}} \Leftrightarrow \boxed{\int}$$

Autonomous LDS Block Diagram (3.3)



Linear Circuit Example (1.2)



$$\begin{bmatrix} i_c \\ v_l \end{bmatrix} = F \begin{bmatrix} v_c \\ i_l \end{bmatrix}$$

$$C \frac{dv_c}{dt} = i_c \qquad L \frac{di_l}{dt} = v_l$$

$$C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_p \end{bmatrix} \qquad L = \begin{bmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_r \end{bmatrix}$$

Linear Circuit Example (2.2)

$$x = \begin{bmatrix} v_c \\ i_L \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix}$$

$$\dot{x} = Ax$$

spice

$$\dot{x} = \underbrace{\begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} F}_{A} x$$

Chemical Reactions (1.3)

chemical reaction involving n chemicals

x_i: concentration of chemical i

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

A ⇔ usually very sparse.

$$A \xrightarrow{k_1} B \xrightarrow{k_2} c$$

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

Chemical Reactions (2.3)

$K_1 = K_2 = 1$

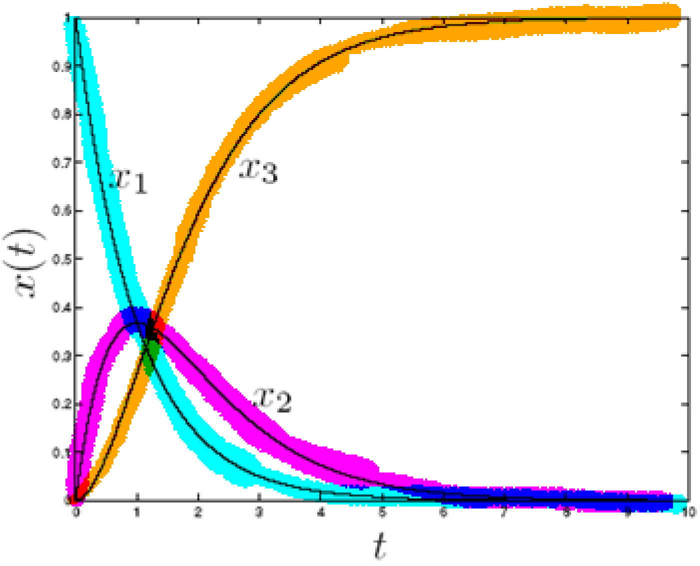
$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A \xrightarrow{1} B \xrightarrow{1} C$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \quad \begin{matrix} - \\ + \end{matrix}$$

$0 \quad 0 \quad 0 \quad 0$

$\dot{x}_2 = x_1 - x_2$



Finite-State Discrete-Time Markov Chain (1.3)

$z(t) \in \{1 \dots n\}$ random sequence with

$$\text{Prob}\,(z(t+1) = i \mid z(t) = j) \triangleq P_{ij}$$

$P \in \mathbb{R}^{n \times n}$ matrix of transition probabilities

jump markov state

$$\text{Prob}\,(z(t) = 1,2,3) = [1\ 1\ 1\ 0 \text{---} 0]\, p(t)$$

$$p(t+1) = P p(t) \rightarrow p(t+1) \neq p(t) P^T$$

Finite-State Discrete-Time Markov Chain (2.3)

P is sparse

nodes are the states

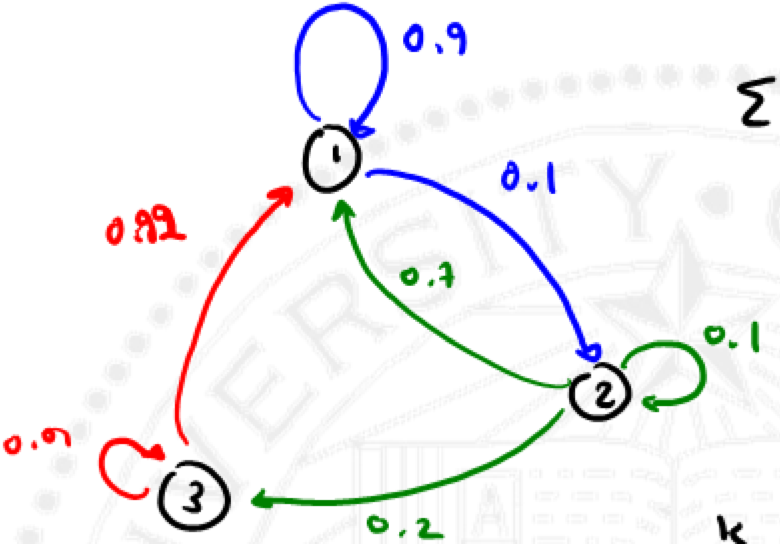
edges are transition probabilities

Finite-State Discrete-Time Markov Chain (3.3)

① "ok"

② "down"

③ under repair



Σ Col P = 1.

$$p_{k+1} = \begin{bmatrix} 0.9 & 0.7 & 0.99 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0.01 \end{bmatrix} p_k$$

$\underline{P^k \; p(0)}$ = end.

Markov chain

Mixing

Numerical Integration of Continuous System (1.3)

compute an approximate solution of $\dot{x} = Ax \quad x(0)=x_0$

small time step : h

x does not change very much over time interval h.

$$x(h+t) \simeq x(t) + h\dot{x}(t) = x(t) + hAx(t)$$

$$= [I+Ah]\,x(t)$$

$$x(kh) \simeq [I+Ah]^k\, x(0) \leftarrow \text{Forwards euler}$$

$$x(t+h) = [I-Ah]^{-1}\, x(t) \leftarrow \text{Backwards euler}$$

"works"

Runge-Kutta

Higher Order LDS (1.3)

$$x^{(k)} = A_{k-1} x^{(k-1)} + \dots + A_1 x^{(1)} + A_0 x \qquad x(t) \in \mathbb{R}^n$$

$\dot{x} = Ax$ overloaded $\quad x^{(n)}$ nth derivative of x

$x^{(0)} = x$

$x^{(1)} = \dot{x}$

$x^{(2)} = \ddot{x}$

$x^{(3)} = \dddot{x}$

$\vdots$

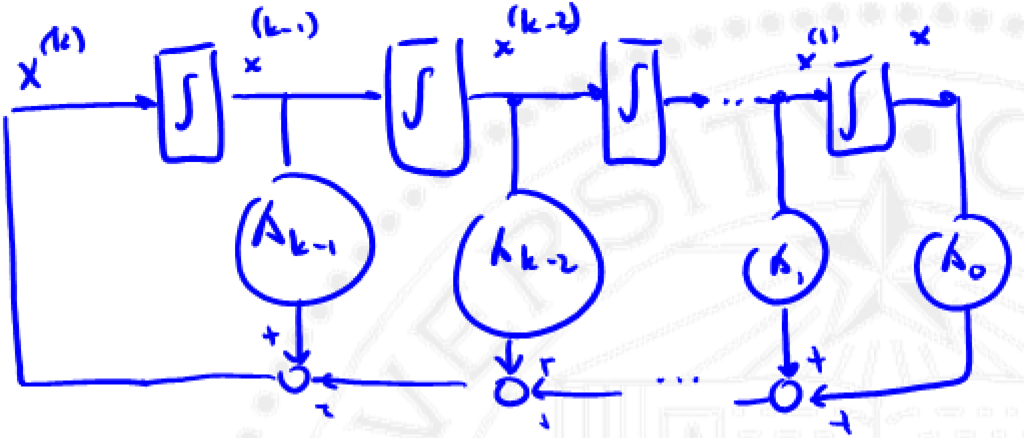
$$z \triangleq \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbb{R}^{nk}$$

shift matrix

$$\dot{z} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & I \\ A_0 & A_1 & \cdots & \cdots & A_{k-1} \end{bmatrix} z$$

← Block companion form.

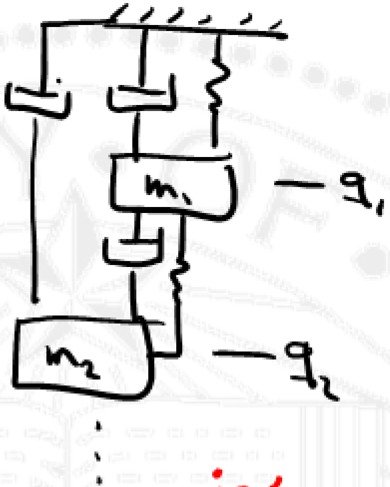
Higher Order LDS (2.3)



chain of integrators

Example: Mechanical Systems

$$M\ddot{q} + D\dot{q} + Kq = 0$$



$q(t) \in \mathbb{R}^k$ generalized displacements

$M \hat{=}$ mass matrix $\left(\begin{smallmatrix}\ddots & & \\ & m & \\ & & \ddots\end{smallmatrix}\right)$

$K \hat{=}$ stiffness matrix

$D \hat{=}$ damping matrix

$\dot{x} = Ax$.

$$x \hat{=} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}}^{A} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Linearization near Equilibrium (1.3)

Non-linear time invariant differential equation.

$\dot{x} = f(x)$ $\quad f : \mathbb{R}^n \to \mathbb{R}^n$

x_e equilibrium point $\quad f(x_e) = 0 \quad x(t) = x_e \ \forall t$.

$x(t)$ near x_e $\quad \dot{x}(t) = f(x(t)) \approx f(x_e) + \underbrace{Df(x_e)}_{\text{Jacobian of } f}(x(t) - x_e)$

$\delta x \triangleq x(t) - x_e$

$\dot{\underline{\delta x}} \approx \underbrace{Df(x_e)}_{F} \underline{\delta x}$

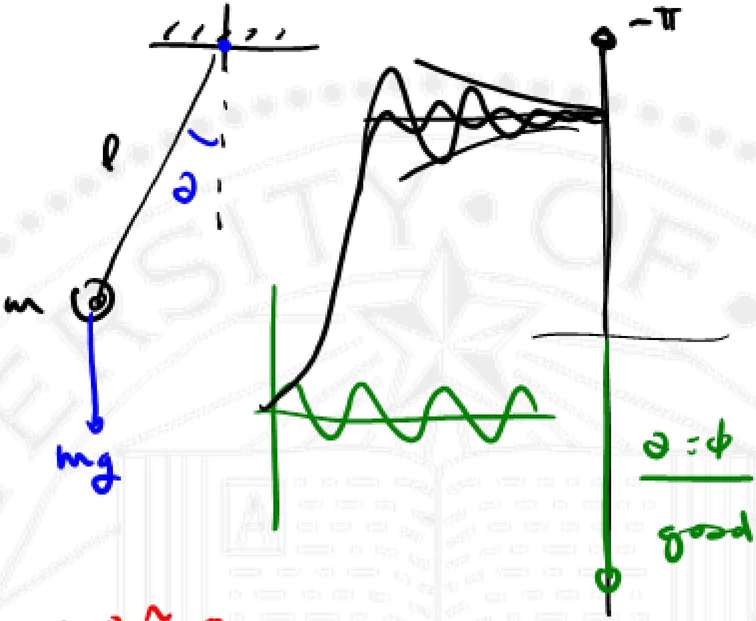
small signal model
linearized dynamics
perturbation equations

Linearization near Equilibrium (2.3)

$$m\ell^2 \ddot{\theta} = -\ell m g \sin\theta$$

$$\dot{\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin\theta \end{bmatrix}$$

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} \delta x$$



$\sin\theta \simeq \theta$.

Does Linearization “work”? (1.2)

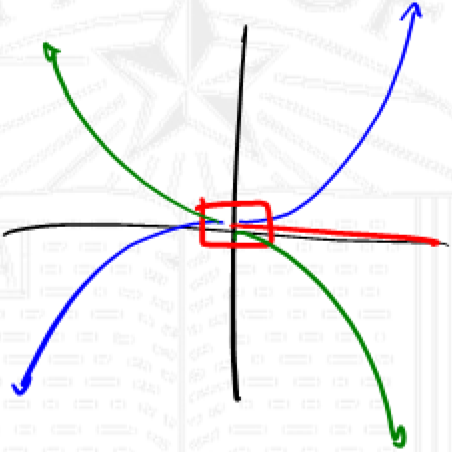
$\dot{x} = -x^3$ $\quad x_e = \phi$. $\quad x(0) > 0$

$x(t) = \sqrt{\frac{1}{x_0^2} + 2t}$

Linearized system: $\delta\dot{x} = \phi$.

$\dot{z} = z^3$ $\quad z(0) > 0$

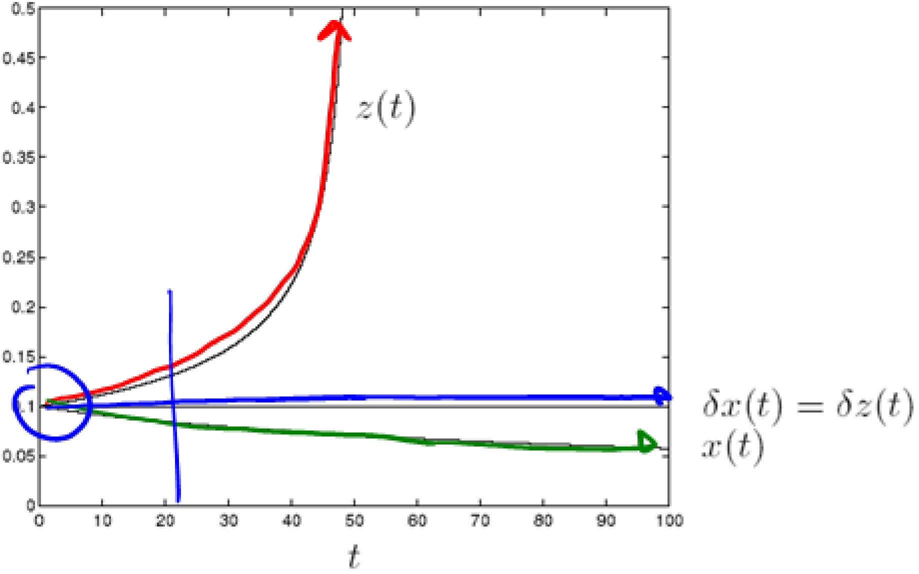
$z(t) = \sqrt{\frac{1}{z_0^2} - 2t}$



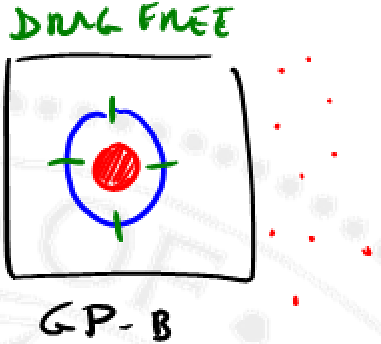
Finite escape time $t = \frac{1}{2z_0^2}$

Does Linearization "work"? (2.2)

$x(0) = z(0) = 0.1$



Linearization along Trajectory (1.3)



$x_{traj} : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$

$\dot{x}_{traj}(t) = f(x_{traj}(t), t)$

$x(t)$ some other trajectory $\dot{x}(t) = f(x(t), t)$

$x(t)$ is near $x_{traj}(t)$

$$\frac{d}{dt}(x - x_{traj}) = f(x,t) - f(x_{traj},t) \simeq D_x f(x_{traj},t)(x - x_{traj})$$

$$\dot{\delta x} \simeq D_x f(x_{traj},t)\, \delta x$$

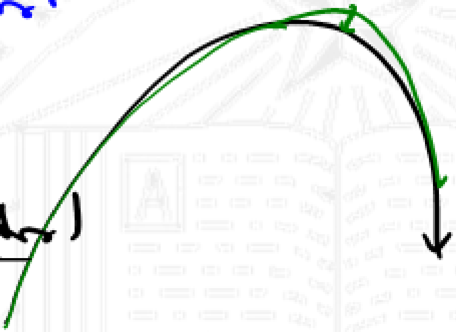
Linearization along Trajectory (2.3)

Linearized or variational trajectory

orbital transfer

optimal control

neighboring optimal control



Linearization along Trajectory (3.3)

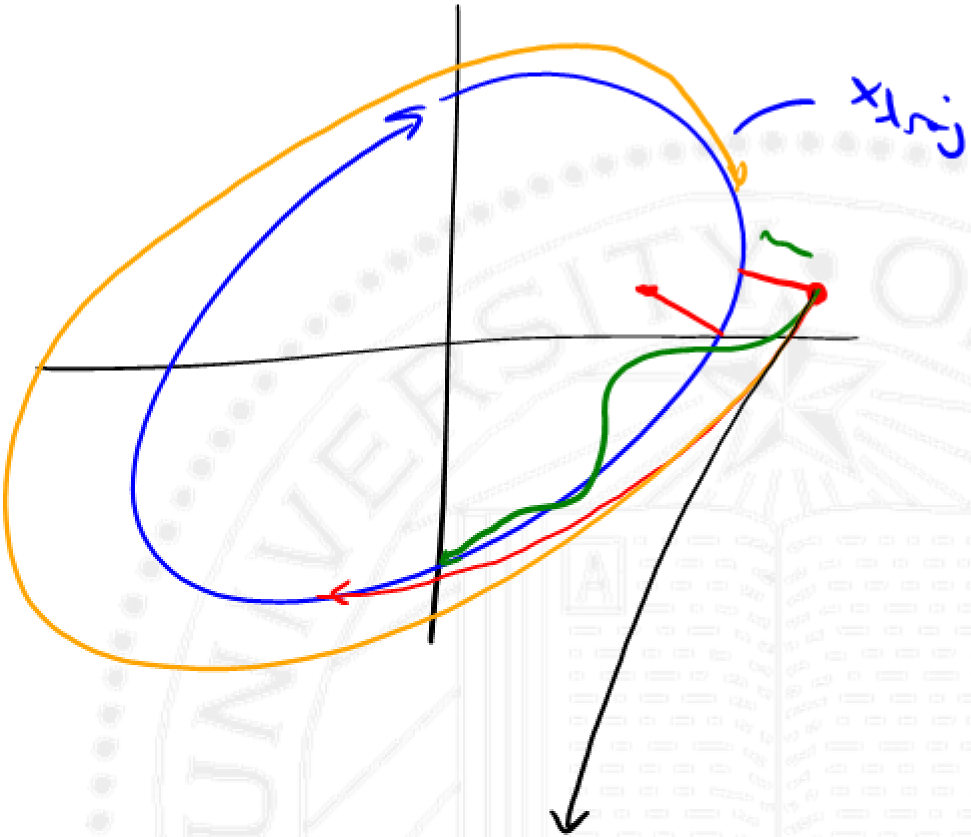
linearized oscillator (Ex)

x_{traj} T-periodic solution <u>NLDE</u>

$\dot{x}_{traj}(t) = f(x_{traj}, t)$ $x_{traj}(t+T) = x_{traj}(t)$

$\delta\dot{x} = A(t)\,\delta x$ $A(t) = Df(x_{traj}, t)$

T-periodic system "Floquet System"


x traj

Questions?

Solution via Laplace Transform and Matrix Exponential

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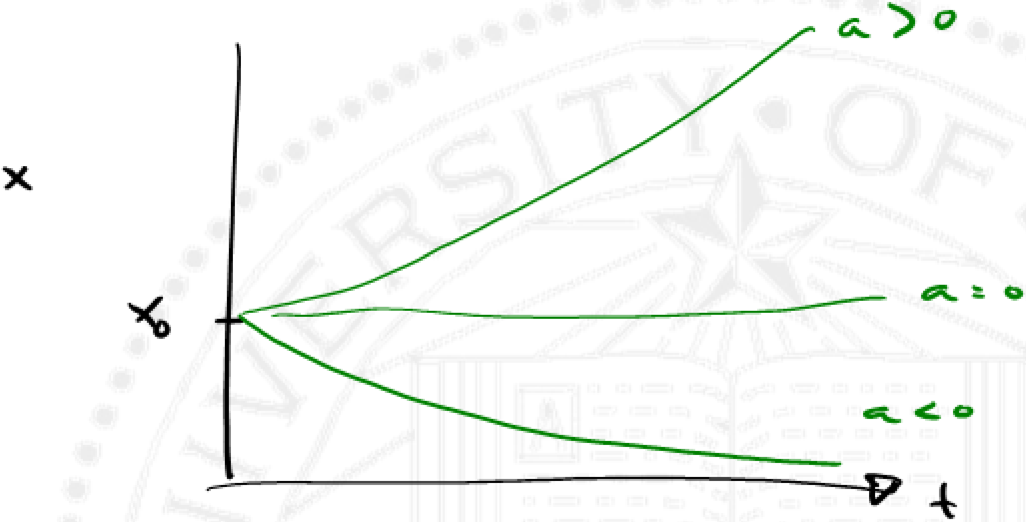
Winter 2016

$\dot{x} = ax \quad \leftarrow$ scalar $\quad x(t) = x_0 e^{at}$



$$\underline{\dot{x}} = A\underline{x} \longrightarrow \underline{x}(t) = \underbrace{e^{At} x_0}$$

Solution via Laplace Transform and Matrix Exponential

- Laplace Tranform
- Solving $x' = Ax$ via Laplace Tranform
- State transition matrix
- Matrix Exponential
- Qualitative behavior and Stability

Laplace Transform of a Matrix (1.3)

$$\mathcal{L}\{x(t)\} \triangleq \int_0^\infty x(t)e^{-st}\,dt \triangleq X(s)$$

$x: \mathbb{R}_+ \to \mathbb{R}$

$X: \mathbb{C} \to \mathbb{C}$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$$

$$\mathcal{L}\{\dot{x} = ax\} = sX(s) - x_0 = aX(s) \to (s-a)X(s) = x_0$$

$$X(s) = \frac{x_0}{s-a}$$

$$\uparrow$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0}{s-a}\right\} = \underline{x_0 e^{at}}$$

Laplace Transform of a Matrix (2.3)

$g : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$ ← trajectory, matrix valued signal

$Z = \mathcal{L}(g)$ where $Z : D \subseteq \underset{\uparrow\ \text{complex}}{\mathbb{C}} \to \mathbb{C}^{p\times q}$

$$Z(s) = \int_0^\infty e^{-st} g(t)\,dt$$

$$Z(s) = \int_0^\infty g(t) e^{-st}\,dt$$

$D \triangleq$ domain of convergence

D includes at least $\{s \mid \mathrm{Re}(s) > a\}$

where a: $|g_{ij}(t)| \le \alpha e^{at}$ $t > 0$ $i = 1..p$ $j = 1..q$