CMPE 240: Introduction to Linear Dynamical Systems (or Linear Operator Theory)

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Overview

- Course Mechanics
- Outline and Topics
- What is a Linear Dynamical System (LDS)
- Why Study Linear Systems
- Some examples

Course Mechanics

- All class info, lectures, homeworks, announcements, etc., on the class webpage at:
  - https://classes.soe.ucsc.edu/cmpe240/Winter16/
- Course Requirements:
- Weekly homeworks (hard)
- Takehome Midterm exam (24 hours, TBD)
- Takehome Final exam (24 hours, TBD)

Prerequisites

- Exposure to Linear Algebra (AMS 27)
- Exposure to Differential Equations and Laplace Transforms
- Not <u>required</u> but useful:
  - Control Systems (CMPE 141/EE 154)
  - Circuit Analysis (EE70)
  - Dynamics and Statics (Physics 5/6)
  - Signals and Systems (EE103)

Books

- Not required for the class, but excellent references:
  - Linear Algebra and its Applications, 3rd Edition, G. Strang (Harcourt, 1988), ISBN: 0155510053 (Abebooks.com ~$7.14 including shipping)
  - Lecture Notes for Stanford's EE263 Class, by Steve Boyd, 2008/2009 (link on website)

Major Topics and Outline

- Linear Algebra and Applications 3-4 week
- Autonomous Linear Dynamical Systems $\dot{x} = Ax$
- Linear Dynamical Systems with Inputs and Outputs — $\dot{x} = Ax + Bu$ $y = Cx + Du$
- Basic Quadratic Control and Estimation

DUAL

Linear Dynamical System (1.3)

Vector Differential Equation

Continuous time linear dynamical system

$$\underbrace{\frac{d\bar{x}}{dt}}_{n\times 1} = \underbrace{A(t)}_{(n\times n)}\underbrace{\bar{x}(t)}_{n\times 1} + B(t)\,\bar{u}(t)$$

$\bar{x} \in \mathbb{R}^n$

$\bar{u}$: $\mathbb{R}^m$ input or control

$$\underbrace{\bar{y}(t)}_{p\times 1,\ \in \mathbb{R}^p} = \underbrace{C(t)}_{p\times n}\,\bar{x}(t) + D(t)\,\bar{u}(t)$$

$t \rightarrow$ time $\in \mathbb{R}$

$\underline{x}(t) \in \mathbb{R}^{n \times 1}$ state $\underline{u}(t) \in \mathbb{R}^{m}$ control/input

Linear Dynamical System (2.3)

$y(t) \in \mathbb{R}^{p}$ output

$A(t) \in \mathbb{R}^{n \times n}$ "Dynamics Matrix"

$B(t) \in \mathbb{R}^{n \times m}$ Input Matrix

$C(t) \in \mathbb{R}^{p \times n}$ Output or Sensor Matrix

$D(t) \in \mathbb{R}^{p \times m}$ Direct Feedthrough Matrix

$$\left[\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right] \longrightarrow \begin{bmatrix} \dot{x} \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ \hline u \end{bmatrix}$$

Linear Dynamical System (3.3)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

STANFORD / CAL TECH 1960's

$$\dot{x} = Fx + Gu$$
$$y = Hx + Ju$$

MIT / HARVARD 1960's

Some LDS Terminology

Most linear systems are LINEAR TIME INVARIANT (LTI)

A, B, C, D not $f(t)$.

$\dot{x} = Ax$ ← autonomous LDS

very often $D = 0^{p \times m}$ ← 0 with dimension

if $u(t)$ and $y(t)$ are both scalars ($m=1, p=1$)

SISO – single input, single output.

MIMO – multiple-input multiple output

$p \neq 1$, $m \neq 1$.

$\underline{x} : \mathbb{R} \to \mathbb{R}^n$

$\underline{x}(1.35)$

Discrete-time LDS

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \text{ } n^{\text{th}} \text{ order diff'l equ'n}$$

$$\underline{x}(t+1) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

$$y(t) = C(t)\,\underline{x}(t) + D(t)\,\underline{u}(t)$$

$$t \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$\left.\begin{aligned} \underline{x}_{k+1} &= A\underline{x}_k + B\underline{u}_k \\ y_k &= C\underline{x}_k + D\underline{u}_k \end{aligned}\right\} \text{ } n^{\text{th}} \text{ order recursion equ'n}$$

$\underline{x} : \mathbb{Z} \to \mathbb{R}^n$

Used to be super-complex, esoteric, not very used

Why Study Linear Systems

mainstream

Applications are everywhere

- Automatic Control Systems
- Signal Processing (Kalman Filter)
- Communications
- Econ / Finance
- Circuit Analysis (SPICE)
- Finite Element Analysis (Mechanical / Civil / Aerospace)
- Aeronautics (CFD)
- GPS – Navigation / Guidance
- Machine Learning / Computer Vision

Usefulness of LDS

Depends on our ability to do computations

Moore's law is on our side.

1980's ~ n = 100.

LAPACK

n = 100,000

Origins and History

Least Squares → goes back to Gauss, Kopernicus 1600s

Linear Algebra ~ 19th century

Classical Optimal Control ~ 19th century

Circuits Signals Systems – 1920's – Bell Labs

First Real Applications – Soviet Price Controls.

Military / Space

Non-linear Dynamical Systems

- Most systems are, in fact, non-linear

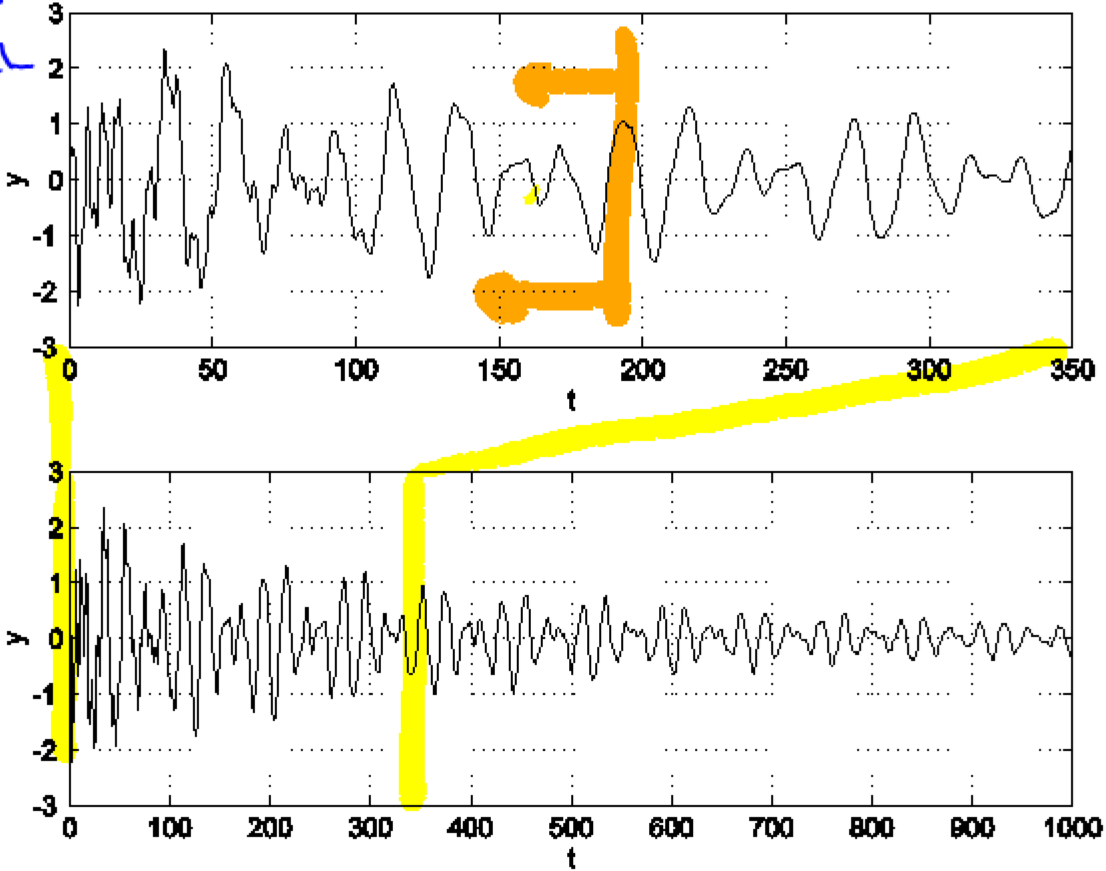
- Most of the techniques for solving non-linear systems are based on linear solutions.

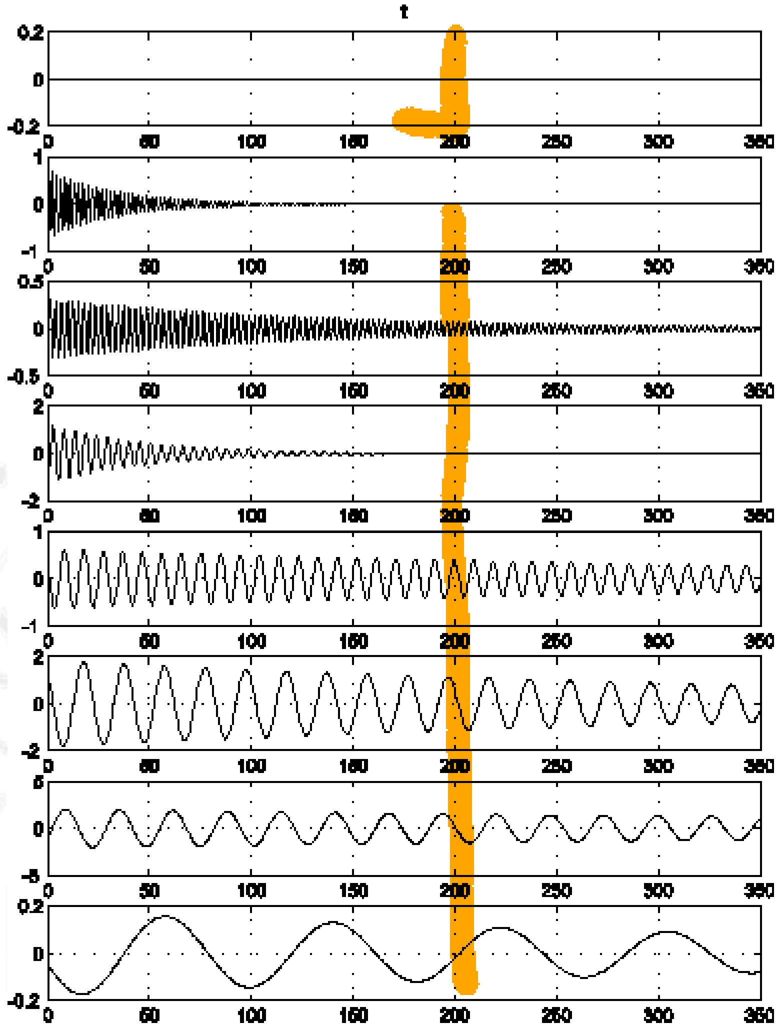
- If you don't understand the linear stuff, you will never understand the non-linear.

n=16

Spectral
factorizal

Examples – Mechanical



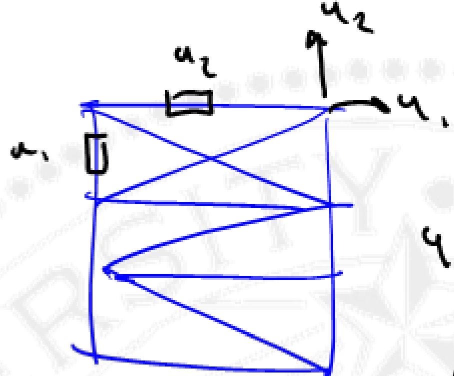

t
0.2
0
-0.2
1
-1
0.5
-0.5
2
-2
5
-5
0
50
100
150
200
250
300
350

Input Design – Mechanical

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$u : \mathbb{R}_+ \to \mathbb{R}^2$



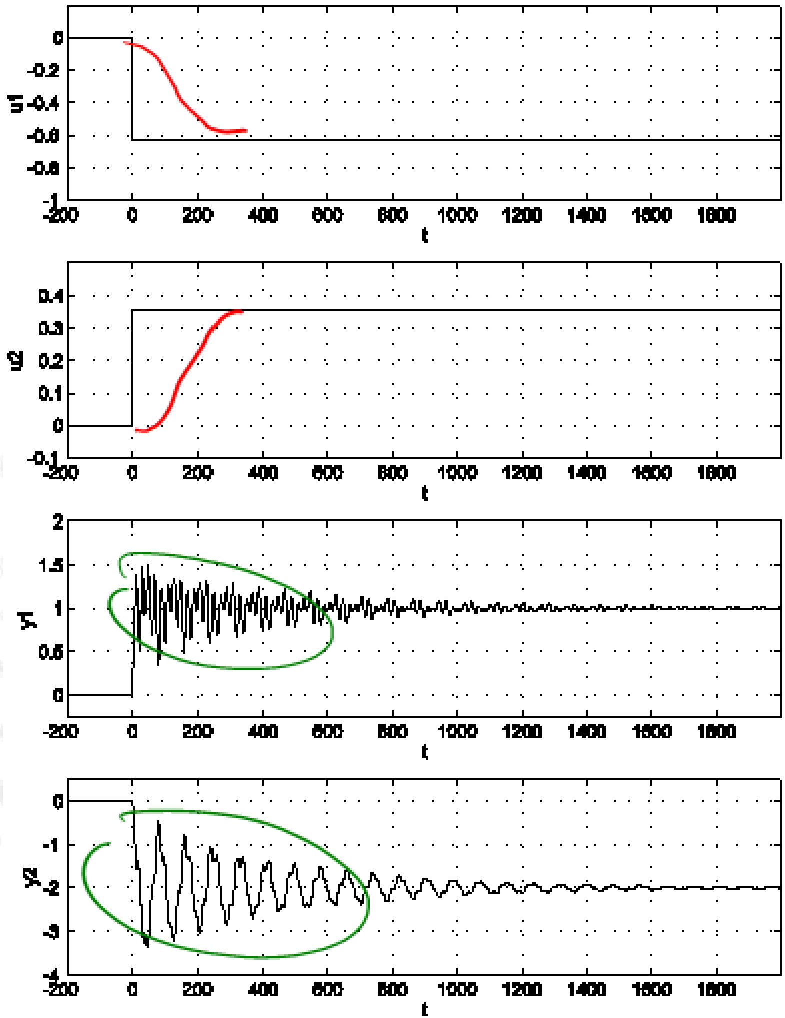
$y(s) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

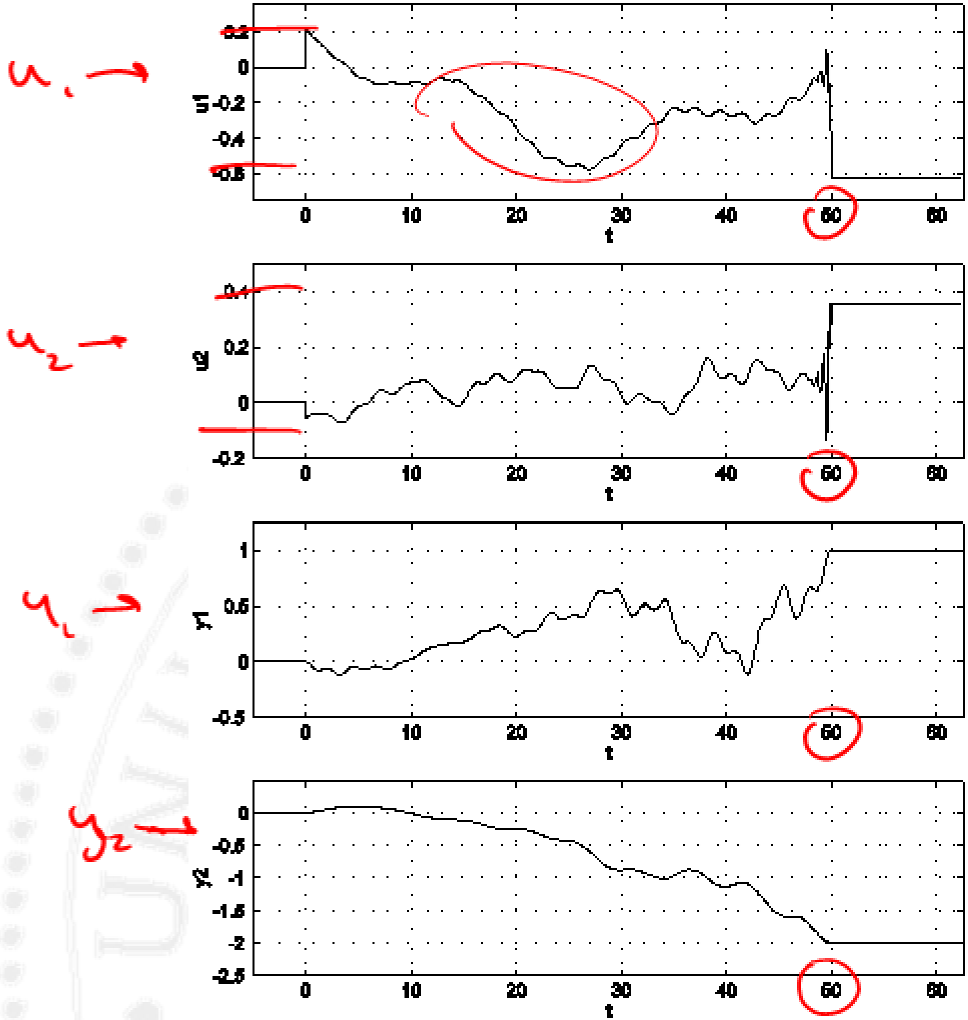
$\cancel{\dot{x}}^{\,0} = Ax + Bu_{ss} \qquad A^{-1}Ax = -A^{-1}Bu_{ss}$

$x_{ss} = -A^{-1}Bu_{ss} \qquad y_{ss} = Cx_{ss} = -CA^{-1}Bu_{ss}$

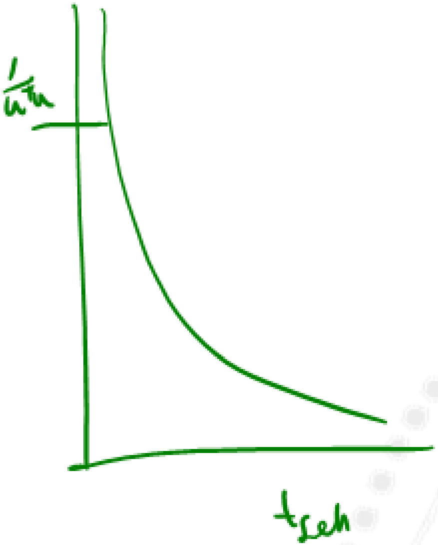
$u_{ss} = -\left[CA^{-1}B\right]^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$

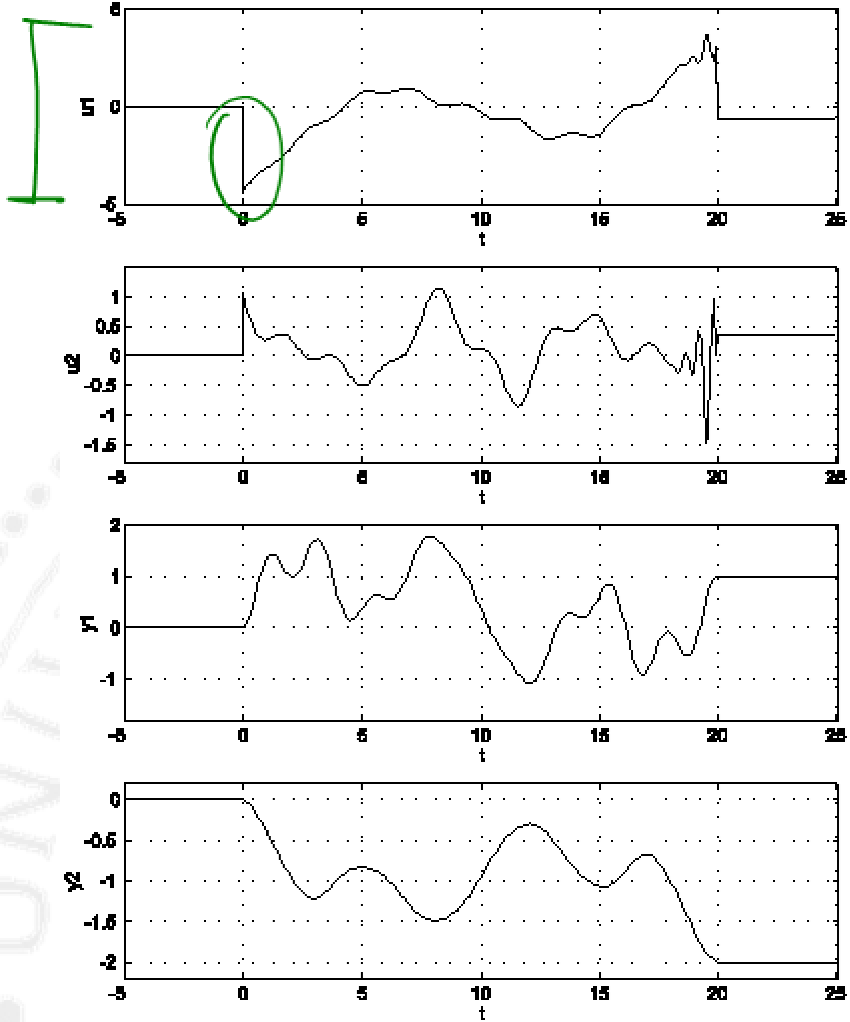
$$u_{r1} = \begin{bmatrix} -0.63 \\ +0.36 \end{bmatrix}$$



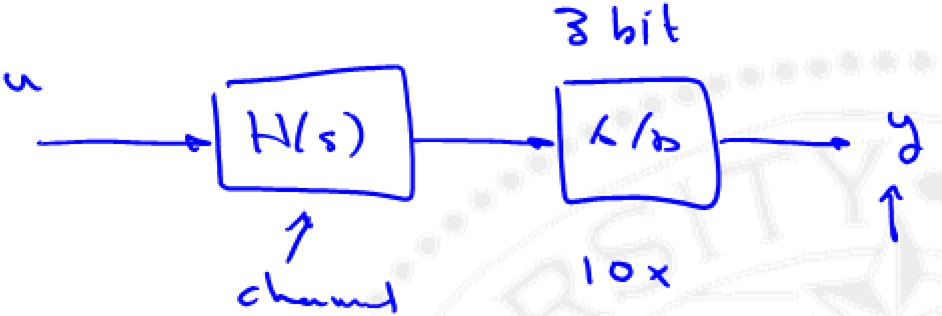


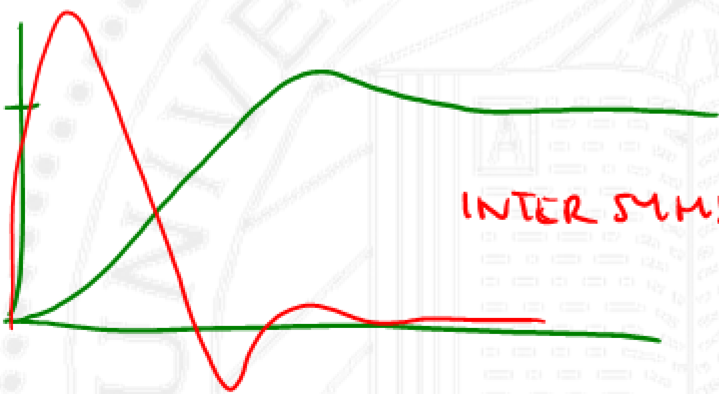
u1
u2
y1
y2
t
0.2
0
-0.2
-0.4
-0.6
0.4
0.2
-0.2
1
0.5
-0.5
-1
-1.5
-2
-2.5
0
10
20
30
40
50
60
u_1
u_2
y_1
y_2

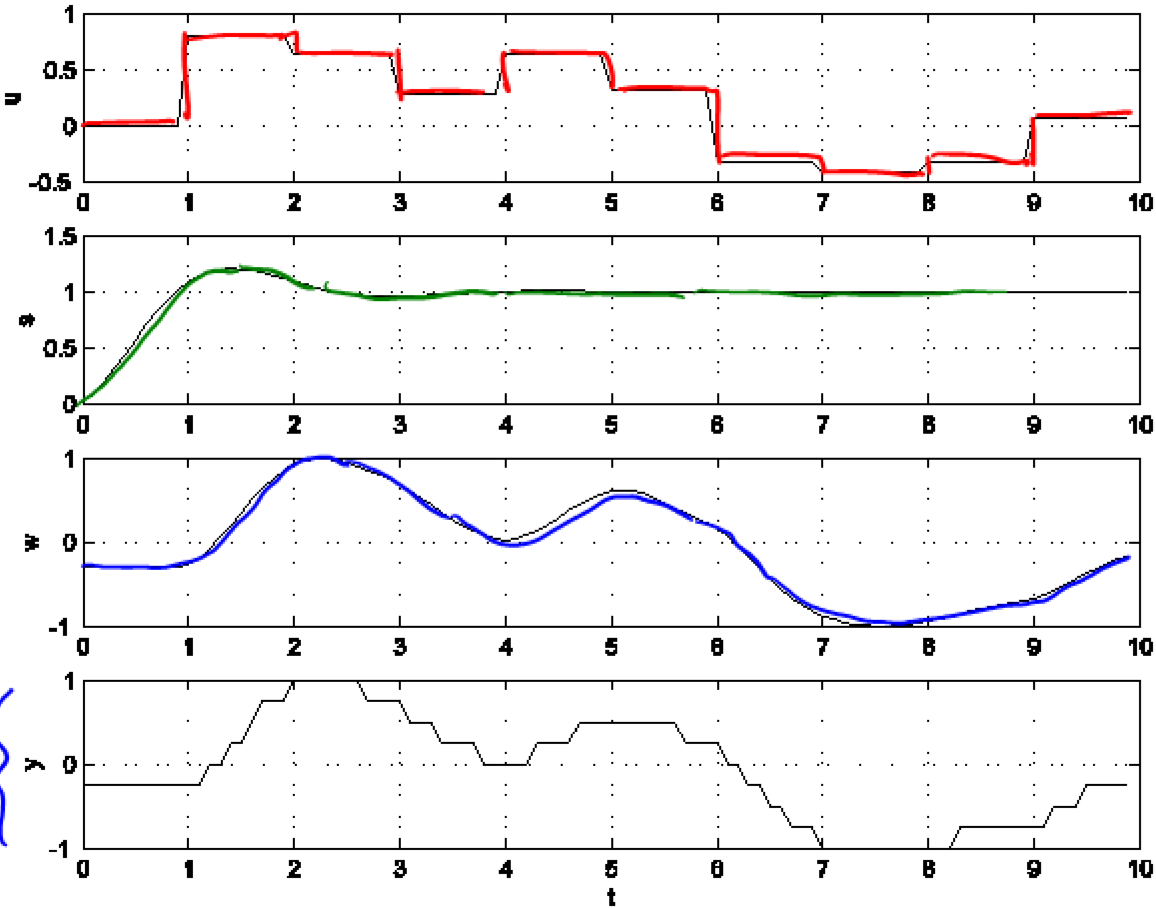



u1
u2
y1
y2
t

Estimation / Filtering





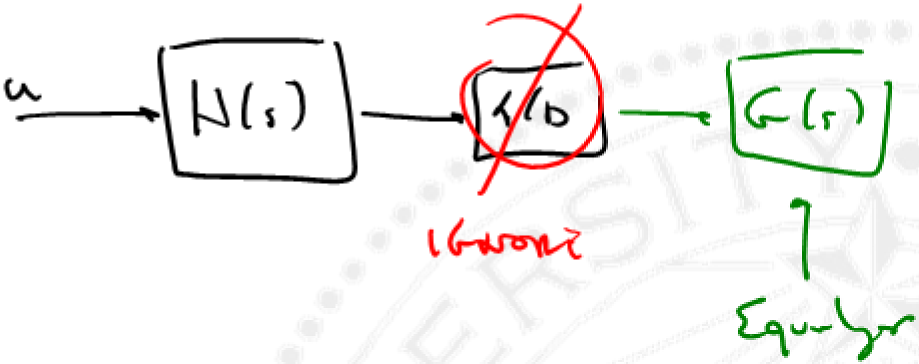


8 steps

Simple Approach

$\frac{0.25\,v}{1}$

+/- 0.125

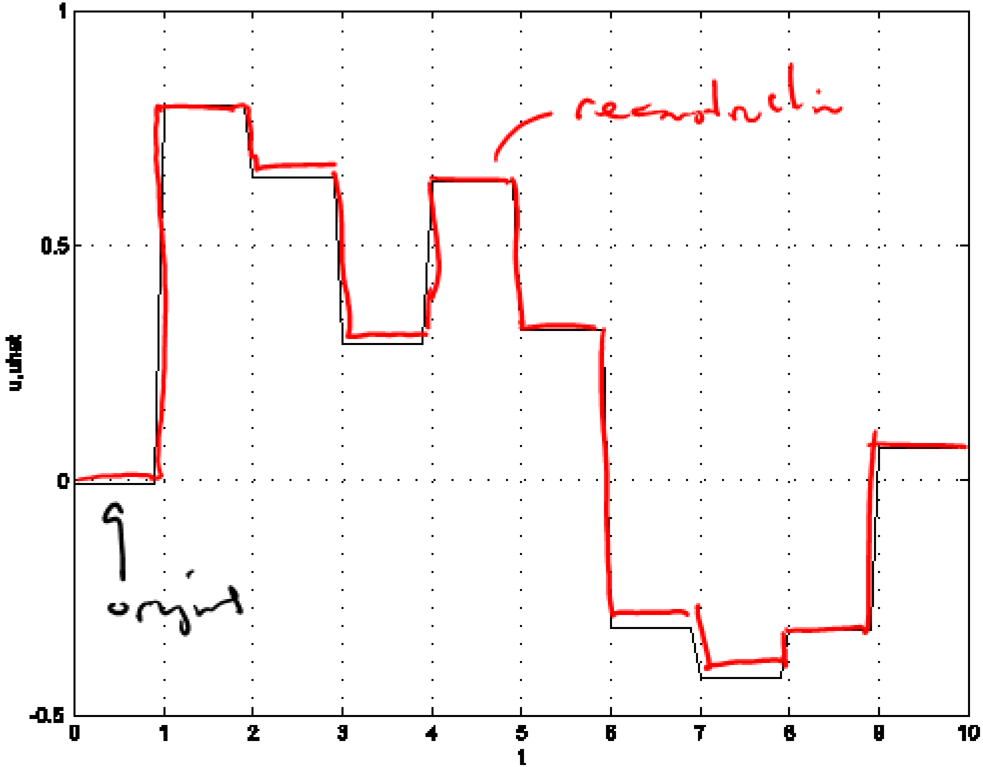


$G(s)H(s) \cong 1$

works <u>TERRIBLY</u>

Estimation Approach

RMS ~ 0.02 V



Questions?

Linear Functions and Examples

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Linear Functions and Examples

- Linear equations and functions
- Engineering Examples
- Interpretations

Linear Equations

m- equations n-variables or unknowns

$$\boxed{y = Ax}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \begin{matrix} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \\ = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{matrix}$$

$$\underset{m\times 1}{Y} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underset{m\times 1}{\underline{y}} = \underset{m\times n}{A} \overset{n\times 1}{\underline{x}}$$

Interpretations

$$y = Ax$$

y is a measurement or observation; x is an unknown to be determined

A maps parameters to measurements

x is an input or action; y is output or the result

A defines a function or transformation that maps $x \in \mathbb{R}^n$ to $y \in \mathbb{R}^m$

$\cancel{x + f(x)}$

Linear Functions

$f : \mathbb{R}^n \to \mathbb{R}^m$ is defined as linear if:

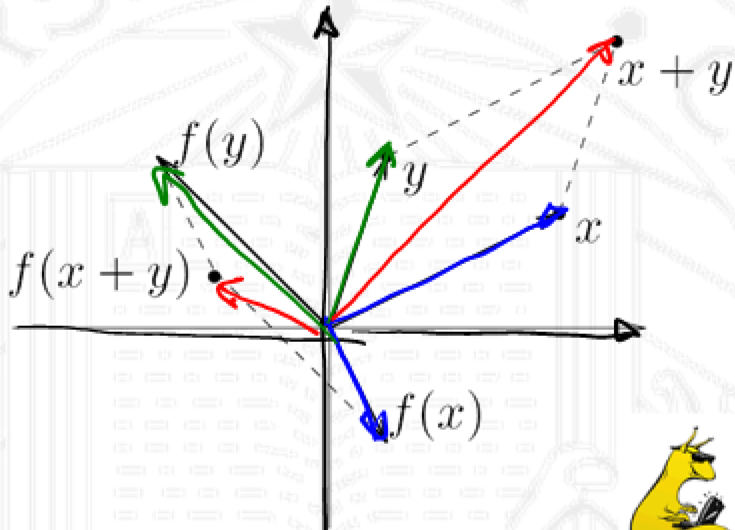
$$\left.\begin{aligned} f(x+y) &= f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}^n \\ f(\alpha x) &= \alpha f(x), \quad \forall\, x \in \mathbb{R}^n,\ \alpha \in \mathbb{R} \end{aligned}\right\} \text{superposition holds.}$$

$\underline{f(x) = Ax}$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = Ax$

$A \in \mathbb{R}^{m \times n}$



Matrix Multiplication function

$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ $\qquad f(x) = Ax \quad A \in \mathbb{R}^{m \times n}$

$f(x+y) = f(x) + f(y) = A(x+y) = Ax + Ay = f(x) + f(y)$

$f(\alpha x) = A\alpha x = \alpha A x = \alpha f(x)$

converse is also true: any linear function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $f(x) = Ax$ for some $A \in \mathbb{R}^{m \times n}$

→ Representation is <u>UNIQUE</u> f: one and only one $f(x) = Ax \ \forall x$.

$y = Ax$ ← generic linear system

Interpretations of *y=Ax*

y is a measurement or an observation
x is unknown to be estimated

ESTIMATION

DUAL

x is input or action
y output or result

CONTROL

$y = Ax$ defines a function or transformation that maps $x \in \mathbb{R}^n$ to $y \in \mathbb{R}^m$

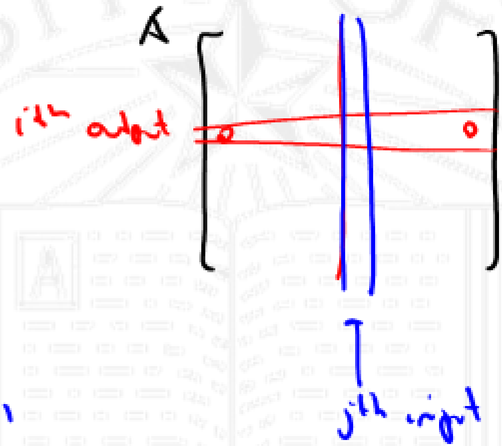
- Robotics
- Forward Model

Interpretations of a_{ij}

$$y = Ax \longrightarrow \quad y_i = \sum_{j=1}^{n} a_{ij} x_j$$

a_{ij} – influence factor

output $\quad$ input



$a_{27} = \phi$

$a_{31} \gg a_{3j} \quad \forall \; j \neq 1$

Interpretations of a_{ij}