University of California, Santa Cruz
Board of Studies in Computer Engineering

CMPE 240: Introduction to Linear Dynamical Systems

Problem Set 6 — Due on 18-Feb-2016

1. *Harmonic oscillator.* The system $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$ is called a *harmonic oscillator.*

   (a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

   (b) Sketch the vector field of the harmonic oscillator.

   (c) The state trajectories describe circular orbits, *i.e.*, $\|x(t)\|$ is constant. Verify this fact using the solution from part (a).

   (d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

2. *Properties of the matrix exponential.*

   (a) Show that $e^{A+B} = e^A e^B$ if A and B commute, *i.e.*, $AB = BA$. The converse is also true, *i.e.*, if $e^{A+B} = e^A e^B$ then A and B commute. (But it is hard to show.)

   (b) Carefully show that $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$.

3. *Determinant of matrix exponential.*

   (a) Suppose the eigenvalues of $A \in \mathbf{R}^{n \times n}$ are $\lambda_1, \ldots, \lambda_n$. Show that the eigenvalues of e^A are $e^{\lambda_1}, \ldots, e^{\lambda_n}$. You can assume that A is diagonalizable, although it is true in the general case.

   (b) Show that $\det e^A = e^{\mathbf{Tr}\, A}$.

   Hint: $\det X$ is the product of the eigenvalues of X, and $\mathbf{Tr}\, Y$ is the sum of the eigenvalues of Y.

4. *Linear system with a quadrant detector.* In this problem we consider the specific system

$$\dot{x} = Ax = \begin{bmatrix} 0.5 & 1.4 \\ -0.7 & 0.5 \end{bmatrix} x.$$

We have a detector or sensor that gives us the sign of each component of the state $x = [x_1\ x_2]^T$ each second:

$$y_1(t) = \text{sgn}(x_1(t)), \quad y_2(t) = \text{sgn}(x_2(t)), \quad t = 0, 1, 2, \ldots$$

where the function $\text{sgn} : \mathbf{R} \to \mathbf{R}$ is defined by

$$\text{sgn}(a) = \begin{cases} 1 & a > 0 \\ 0 & a = 0 \\ -1 & a < 0 \end{cases}$$

There are several ways to think of these sensor measurements. You can think of $y(t) = [y_1(t)\ y_2(t)]^T$ as determining which quadrant the state is in at time t (thus the name *quadrant detector*). Or, you can think of $y(t)$ as a one-bit quantized measurement of the state at time t.

Finally, the problem. You observe the sensor measurements

$$y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Based on these measurements, what values could $y(2)$ possibly take on?

In terms of the quadrants, the problem can be stated as follows. $x(0)$ is in quadrant IV, and $x(1)$ is also in quadrant IV. The question is: which quadrant(s) can $x(2)$ possibly be in?

You do not know the initial state $x(0)$.

Of course, you must completely justify and explain your answer.

5. *Some basic properties of eigenvalues.* Show that

   (a) the eigenvalues of A and A^T are the same

   (b) A is invertible if and only if A does not have a zero eigenvalue

   (c) if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and A is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$,

   (d) the eigenvalues of A and $T^{-1}AT$ are the same.

   Hint: you'll need to use the facts $\det AB = \det A \det B$ and $\det A^{-1} = 1/\det A$ (provided A is invertible).

6. *Characteristic polynomial.* Consider the characteristic polynomial $\mathcal{X}(s) = \det(sI - A)$ of the matrix $A \in \mathbf{R}^{n\times n}$.

   (a) Show that $\mathcal{X}$ is *monic*, which means that its leading coefficient is one: $\mathcal{X}(s) = s^n + \cdots$.

   (b) Show that the s^{n-1} coefficient of $\mathcal{X}$ is given by $-\mathbf{Tr}\,A$. ($\mathbf{Tr}\,X$ is the *trace* of a matrix: $\mathbf{Tr}\,X = \sum_{i=1}^n X_{ii}$.)

(c) Show that the constant coefficient of $\mathcal{X}$ is given by $\det(-A)$.

(d) Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A, so that

$$\mathcal{X}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n).$$

By equating coefficients show that $a_{n-1} = -\sum_{i=1}^n \lambda_i$ and $a_0 = \prod_{i=1}^n (-\lambda_i)$.

7. *Left eigenvector properties.* Suppose w is a left eigenvector of $A \in \mathbf{R}^{n\times n}$ with real negative eigenvalue λ.

(a) Find a simple expression for $w^T e^{At}$.

(b) Let $\alpha < \beta$. The set $\{ z \mid \alpha \le w^T z \le \beta \}$ is referred to as a *slab*. Briefly explain this terminology. Draw a picture in $\mathbf{R}^2$.

(c) Show that the slab $\{ z \mid 0 \le w^T z \le \beta \}$ is invariant for $\dot{x} = Ax$.

8. *Some Matlab exercises.* Consider the continuous-time system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -0.1005 & 1.0939 & 2.0428 & 4.4599 \\ -1.0880 & -0.1444 & 5.9859 & -3.0481 \\ -2.0510 & -5.9709 & -0.1387 & 1.9229 \\ -4.4575 & 3.0753 & -1.8847 & -0.1164 \end{bmatrix}.$$

(a) What are the eigenvalues of A? Is the system stable? You can use the command `eig` in Matlab.

(b) Plot a few trajectories of $x(t)$, *i.e.*, $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$, for a few initial conditions. To do this you can use the matrix exponential command in Matlab `expm` (*not* `exp` which gives the element-by-element exponential of a matrix), or more directly, the Matlab command `initial` (use `help initial` for details.) Verify that the qualitative behavior of the system is consistent with the eigenvalues you found in part (8a).

(c) Find the matrix Z such that $Zx(t)$ gives $x(t + 15)$. Thus, Z is the '15 seconds forward predictor matrix'.

(d) Find the matrix Y such that $Yx(t)$ gives $x(t - 20)$. Thus Y reconstructs what the state was 20 seconds ago.

(e) Briefly comment on the size of the elements of the matrices Y and Z.

(f) Find $x(0)$ such that $x(10) = [1\ 1\ 1\ 1]^T$.