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BOARD OF STUDIES IN COMPUTER ENGINEERING

CMPE 240: INTRODUCTION TO LINEAR DYNAMICAL SYSTEMS

GABRIEL HUGH ELKAIM SPRING 2017

Basic Notation

Basic set notation

$\{a_1, \ldots, a_r\}$	the set with elements $a_1, \ldots, a_r$.
$a \in S$	a is in the set S.
$S = T$	the sets S and T are equal, *i.e.*, every element of S is in T and every element of T is in S.
$S \subseteq T$	the set S is a subset of the set T, *i.e.*, every element of S is also an element of T.
$\exists a \in S\ \mathcal{P}(a)$	there exists an a in S for which the property $\mathcal{P}$ holds.
$\forall x \in S\ \mathcal{P}(a)$	property $\mathcal{P}$ holds for every element in S.
$\{\ a \in S \mid \mathcal{P}(a)\ \}$	the set of all a in S for which $\mathcal{P}$ holds (the set S is sometimes omitted if it can be determined from context.)
$A \cup B$	union of sets, $A \cup B = \{\ x \mid x \in A \text{ or } x \in B\ \}$.
$A \cap B$	intersection of sets, $A \cap B = \{\ x \mid x \in A \text{ and } x \in B\ \}$
$A \times B$	Cartesian product of two sets, $A \times B = \{\ (a, b) \mid a \in A,\ b \in B\ \}$

Some specific sets

$\mathbf{R}$	the set of real numbers.
$\mathbf{R}^n$	the set of real n-vectors ($n \times 1$ matrices).
$\mathbf{R}^{1\times n}$	the set of real n-row-vectors ($1 \times n$ matrices).
$\mathbf{R}^{m\times n}$	the set of real $m \times n$ matrices.
j	can mean $\sqrt{-1}$, in the company of electrical engineers.
i	can mean $\sqrt{-1}$, for normal people; i is the polite term in mixed company (*i.e.*, when non-electrical engineers are present.)
$\mathbf{C}$, $\mathbf{C}^n$, $\mathbf{C}^{m\times n}$	the set of complex numbers, complex n-vectors, complex $m \times n$ matrices.
$\mathbf{Z}$	the set of integers: $\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$.
$\mathbf{R}_+$	the nonnegative real numbers, *i.e.*, $\mathbf{R}_+ = \{\ x \in \mathbf{R} \mid x \geq 0\ \}$.
$[a, b]$, $(a, b]$, $[a, b)$, (a, b)	the real intervals $\{\ x \mid a \leq x \leq b\ \}$, $\{\ x \mid a < x \leq b\ \}$, $\{\ x \mid a \leq x < b\ \}$, and $\{\ x \mid a < x < b\ \}$, respectively.

Vectors and matrices

We use square brackets [and] to construct matrices and vectors, with white space delineating the entries in a row, and a new line indicating a new row. For example [1 2] is a row vector in $\mathbf{R}^{2\times 1}$, and $\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$ is matrix in $\mathbf{R}^{2\times 3}$. $[1\ 2]^T$ denotes a column vector, *i.e.*, an element of $\mathbf{R}^{2\times 1}$, which we abbreviate as $\mathbf{R}^2$.

We use curved brackets (and) surrounding lists of entries, delineated by commas, as an alternative method to construct (column) vectors. Thus, we have three ways to denote a column vector:

$$(1,2) = [1\ 2]^T = \left[\begin{array}{c} 1 \\ 2 \end{array}\right].$$

Note that in our notation scheme (which is fairly standard), $[1,2,3]$ and (1 2 3) aren't used.

Functions

The notation $f : A \to B$ means that f is a function on the set A into the set B. The notation $b = f(a)$ means b is the value of the function f at the point a, where $a \in A$ and $b \in B$. The set A is called the *domain* of the function f; it can thought of as the set of legal parameter values that can be passed to the function f. The set B is called the *codomain* (or sometimes range) of the function f; it can thought of as a set that contains all possible returned values of the function f.

There are several ways to think of a function. The formal definition is that f is a subset of $A \times B$, with the property that for every $a \in A$, there is exactly one $b \in B$ such that $(a,b) \in f$. We denote this as $b = f(a)$.

Perhaps a better way to think of a function is as a *black box* or (software) *function* or *subroutine*. The domain is the set of all legal values (or data types or structures) that can be passed to f. The codomain of f gives the data type or data structure of the values returned by f.

Thus $f(a)$ is *meaningless* if $a \notin A$. If $a \in A$, then $b = f(a)$ is an element of B. Also note that the *function* is denoted f; it is *wrong* to say 'the function $f(a)$' (since $f(a)$ is an element of B, not a function). Having said that, we do sometimes use sloppy notation such as 'the function $f(t) = t^3$'. To say this more clearly you could say 'the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(t) = t^3$ for $t \in \mathbf{R}$'.

Examples

- $-0.1 \in \mathbf{R}$, $\sqrt{2} \in \mathbf{R}_+$, $1-2j \in \mathbf{C}$ (with $j=\sqrt{-1}$).

- The matrix
$$A = \begin{bmatrix} 0.3 & 6.1 & -0.12 \\ 7.2 & 0 & 0.01 \end{bmatrix}$$
is an element in $\mathbf{R}^{2\times 3}$. We can define a function $f : \mathbf{R}^3 \to \mathbf{R}^2$ as $f(x) = Ax$ for any $x \in \mathbf{R}^3$. If $x \in \mathbf{R}^3$, then $f(x)$ is a particular vector in $\mathbf{R}^2$. We can say 'the function f is linear'. To say 'the function $f(x)$ is linear' is technically wrong since $f(x)$ is a vector, not a function. Similarly we can't say 'A is linear'; it is just a matrix.

- We can define a function $f : \{a \in \mathbf{R} \mid a \neq 0\} \times \mathbf{R}^n \to \mathbf{R}^n$ by $f(a,x) = (1/a)x$, for any $a \in \mathbf{R}$, $a \neq 0$, and any $x \in \mathbf{R}^n$. The function f could be informally described as division of a vector by a nonzero scalar.

- Consider the set $A = \{\ 0,\ -1,\ 3.2\ \}$. The elements of A are 0, -1 and 3.2. Therefore, for example, $-1 \in A$ and $\{\ 0,\ 3.2\ \} \subseteq A$. Also, we can say that $\forall x \in A,\ -1 \leq x \leq 4$ or $\exists x \in A,\ x > 3$.

- Suppose $A = \{\ 1,\ -1\ \}$. Another representation for A is $A = \{\ x \in \mathbf{R} \mid x^2 = 1\ \}$.

- Suppose $A = \{\ 1,\ -2,\ 0\ \}$ and $B = \{\ 3,\ -2\ \}$. Then
$$A \cup B = \{\ 1,\ -2,\ 0,\ 3\ \}, \quad A \cap B = \{\ -2\ \}.$$

- Suppose $A = \{\ 1,\ -2,\ 0\ \}$ and $B = \{1, 3\}$. Then
$$A \times B = \{\ (1,1),\ (1,3),\ (-2,1),\ (-2,3),\ (0,1),\ (0,3)\ \}.$$

- $f : \mathbf{R} \to \mathbf{R}$ with $f(x) = x^2 - x$ defines a function from $\mathbf{R}$ to $\mathbf{R}$ while $u : \mathbf{R}_+ \to \mathbf{R}^2$ with
$$u(t) = \begin{bmatrix} t\cos t \\ 2e^{-t} \end{bmatrix}.$$
defines a function from $\mathbf{R}_+$ to $\mathbf{R}^2$.