MIDTERM REVIEW

01/NOV/2007

Topics

Linear Functions — $f: \mathbb{R}^n \to \mathbb{R}^m$

superposition holds $\begin{cases} f(a+b) = f(a) + f(b) \\ f(\alpha a) = \alpha f(a) \end{cases}$

$f \longrightarrow y = Ax \quad y \in \mathbb{R}^m, \; x \in \mathbb{R}^n.$

Interpretations –

A_{ij}

input

$C = AB$

Dual — Estimation
Control / Design

Vector Spaces
Independence
Basis
Dimension
Norms
Angles

Inner Products
"onto"
Range
Rank
Null
Span

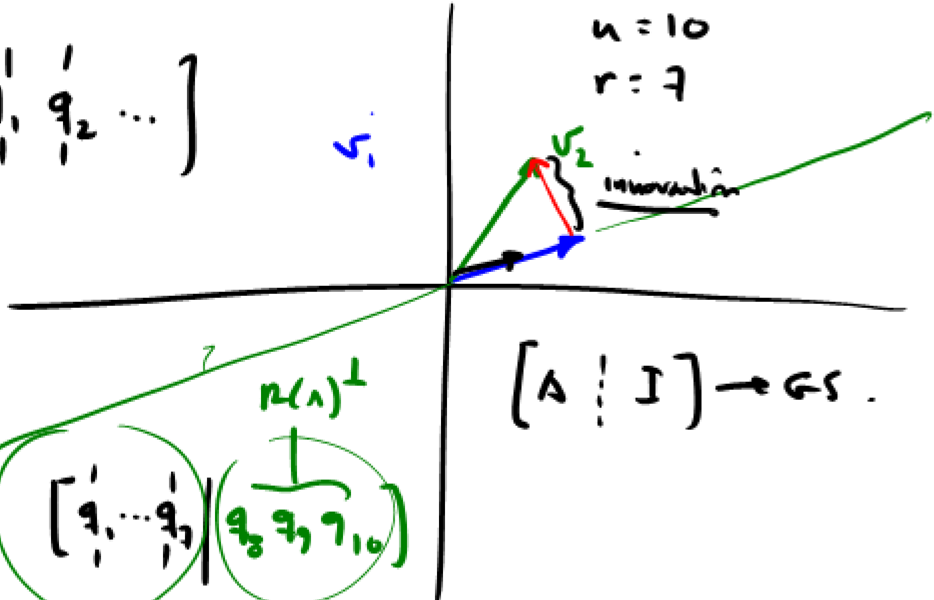
$\text{Range}\{V\} \quad V = [v_1 \; v_2]$

$$\text{span}\{v_1, v_2\} = \alpha v_1 + \beta v_2 \quad \forall \alpha, \beta \in \mathbb{R}$$

$$\begin{bmatrix} | & | & \\ q_1 & q_2 & \cdots \\ | & | & \end{bmatrix}$$

$n = 10$

$r = 7$



$[A \,\vdots\, I] \rightarrow GS.$

$R(A)^{\perp}$

$$\left[\begin{array}{ccc|ccc} | & & | & & & \\ q_1 & \cdots & q_7 & q_8 & q_9 & q_{10} \\ | & & | & & & \end{array}\right]$$

- Left / right inverse

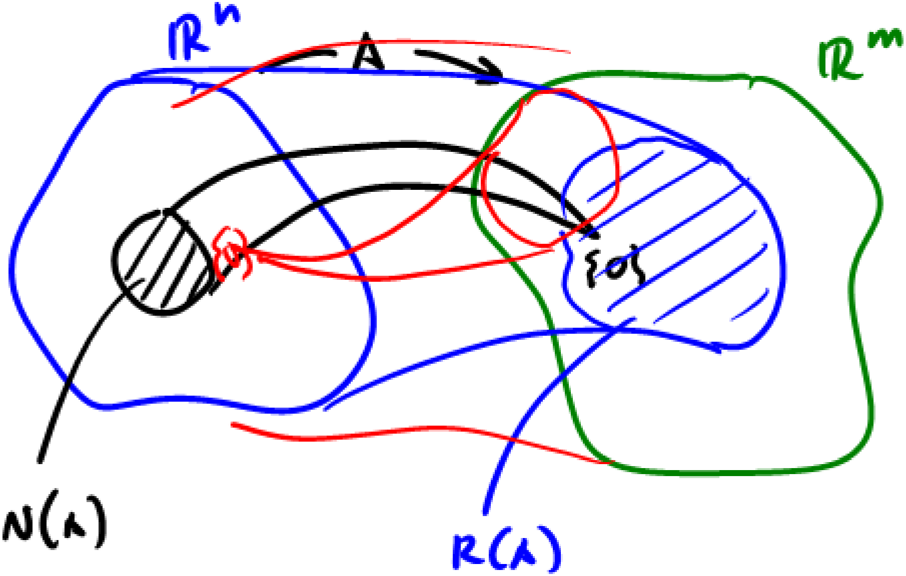
- Orthonormality
  - orthonormal sets of vectors
  - orthogonal matrix ($A^{-1} = A^T$).

- Rotations / Reflections (det $= \pm 1$).

- QR factorization R^{-1} exists

- Gram-Schmidt Procedure

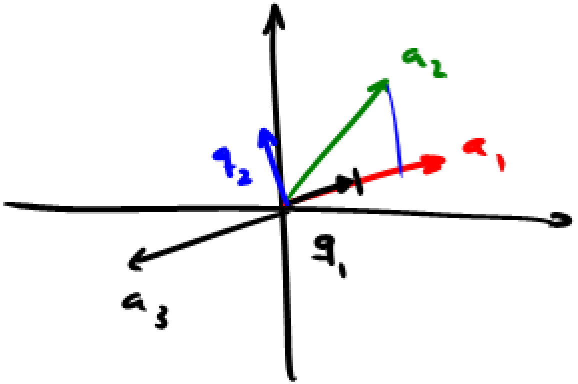
- Decompose a matrix $A = QR$



$$\boxed{R(A) \overset{\perp}{+} N(A^T) = \mathbb{R}^m}$$

$$R(A) \perp N(A^T) \rightarrow \mathbb{R}^m$$

$$R(A^T) \perp N(A) \rightarrow \mathbb{R}^n$$


a2
q2
a1
q1
a3

LEAST SQUARES

ESTIMATION

noise

$$y = Ax + v$$

parameters

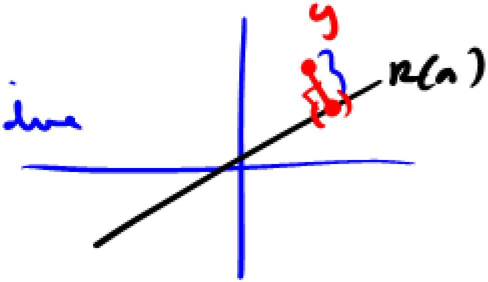
measurement

$$x_{ls} = (A^TA)^{-1}A^Ty$$

$$A^{+} \triangleq (A^TA)^{-1}A^T$$ pseudo-inverse

Moore-Penrose inverse of A

- Projection
- Estimation
- Orthogonality of the Residual
- Geometric Interpretation

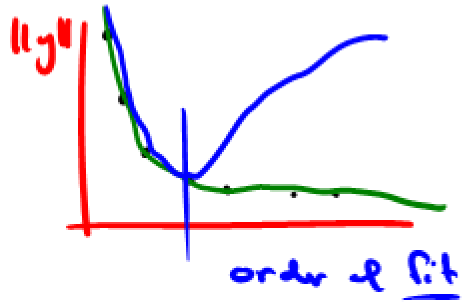


BLUE — Best Linear Unbiased Estimator.

$y = Ax + v$ $\quad \hat{x} = x_{ls} = x$ if $v = \phi$.

LEAST SQUARES APPLICATIONS

- Growing sets of Regressors (polynomial fitting)
- System Identification
  - Cross Validation

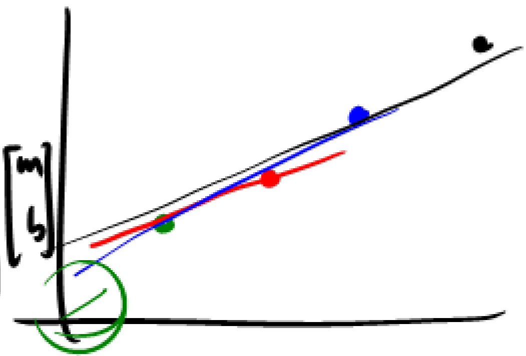


- Growing sets of measurements

  FAST UPDATE RLS

$y = mx + b$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$



Extensions to Least Squares

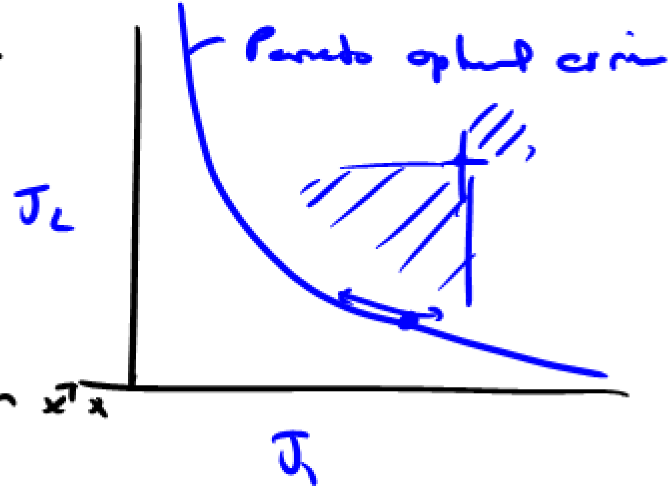
- Regularized Least Squares (Tychonov Regularization)

$(A^T A + \mu I)^{-1} A^T y$. ← changes is full rank

- Non-linear least squares
  - linearize about a point
  - Gauss-Newton method

Mixed optimum criteria

$J_1 + \mu J_2$



Solve $y = Ax$ with min $x^T x$

min $x^T x$ underdetermined set (A - fat)

subj $Ax - y = 0$.

$x_{ln} = A^T (AA^T)^{-1} y$ right inverse of A.

Lagrange Multipliers

$$L = x^T x + \lambda (Ax - y)$$

$$\left.\begin{array}{l} \nabla_x L = 0 \\ \nabla_\lambda L = 0 \end{array}\right] \rightarrow \text{find solution}$$

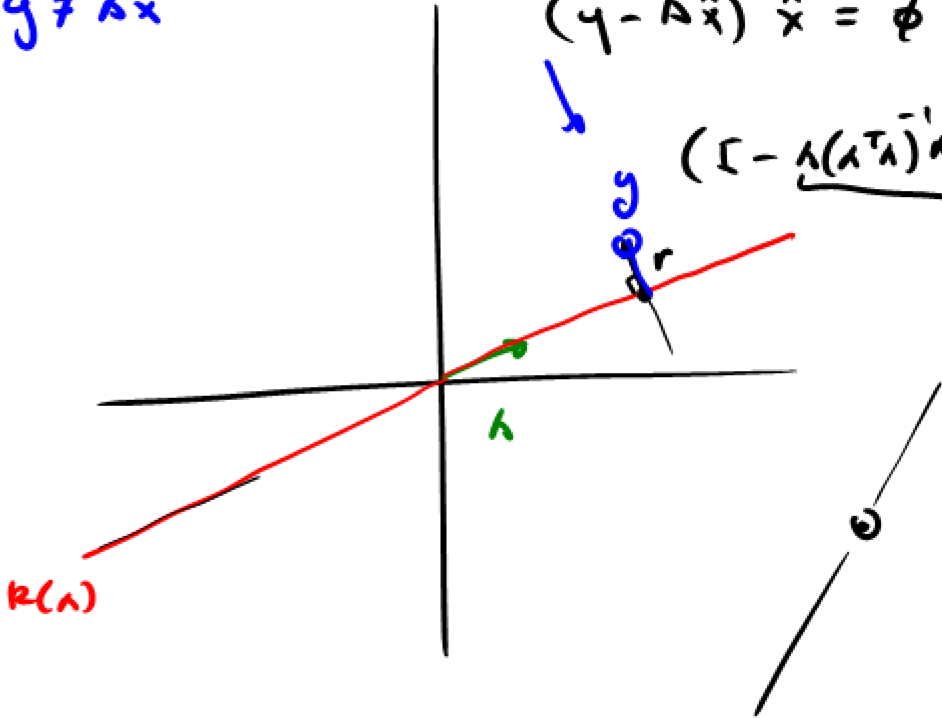
$$\left[\begin{array}{ll} \min & \|Cx + D\| \\ \text{subj} & Ax - y \end{array}\right)$$

$$\longrightarrow \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^T y \\ D \end{bmatrix}$$

$y \neq Ax$

$$(y - A\hat{x})^T \hat{x} = \phi$$

$$\underbrace{\left(I - A(A^TA)^{-1}A^T\right)} y = \underline{r}$$

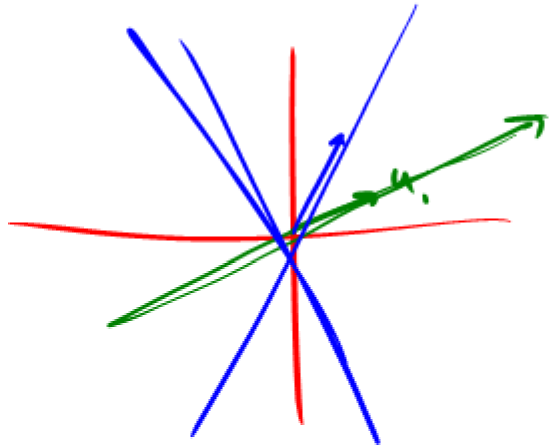


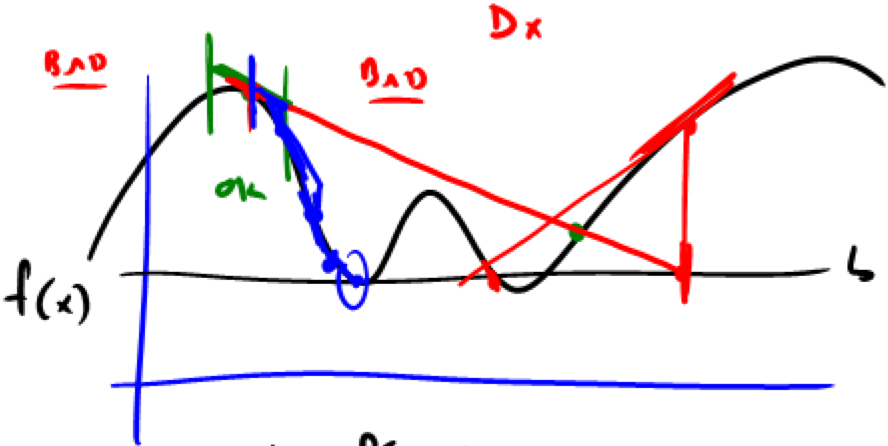
$$(I - u_I u_I^T)$$

$u_i \quad u_i u_i^T$ = rank (1) matrix.

$I - u_i u_i^T \rightarrow$

$$\left[I - \sum_{i=1}^{n} u_i u_i^T \right]$$

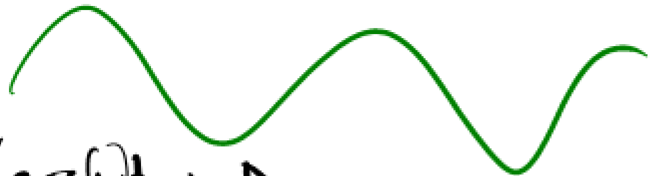




$$\min_{x} f(x) - b$$

$$y = A\sin(\omega t + \phi) + D + \nu$$

IEEE 1057



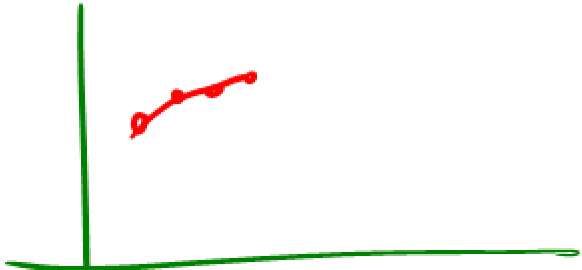
$$y = \underbrace{\hat{A}\sin\omega t + \tilde{B}\cos\omega t}_{\partial F/\partial \omega} + D$$

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} \sin\omega t_1 & \cos\omega t_1 & 1 \\ \vdots & \vdots & \vdots \\ \sin\omega t_n & \cos\omega t_n & 1 \end{bmatrix}}_{} \begin{bmatrix} \hat{A} \\ \tilde{B} \\ D \end{bmatrix}$$

$A(\omega)$

$r(\omega)$

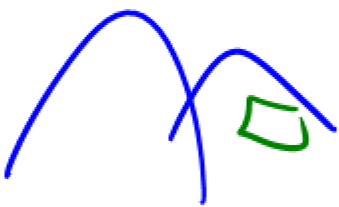
$$\omega_0 \rightarrow \omega_0 + \delta\omega \cdot \left(\left(\frac{\partial F}{\partial \omega}\right)^T\left(\frac{\partial F}{\partial \omega}\right) + \mu I\right)^{-1} \partial F^T$$



$$\begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}$$

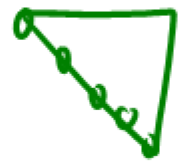
$$\|q\| = 1$$

$$\begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix}$$



SGI

$$\begin{cases} F_{ij} = 1 & \text{if } j \in \underline{Fo}(i) \\ F_{ij} = \phi & \text{otherwise} \end{cases}$$



$$d_i = \beta_i + \gamma_i C_i^{load}/x_i$$

$$d_i = T$$

$$C_i^{load} = C_i^{ext} + \sum_{j \in Fo} C_j^{in}$$

$$T = d_i = \beta_i + \gamma_i \frac{C_i^{load}}{x_i} =$$

$$T = \beta_i + \gamma_i \left[\frac{c_i^{ext} + \sum_{j \in FO(i)} c_j^{in}}{x_i} \right] \qquad c_i^{in} = \alpha_i x_i$$

$$= \beta_i + \gamma_i \frac{\left[c_i^{ext} + \sum_{j \in FO(i)} c_j^{in} \right]}{x_i}$$

$$= \beta_i + \gamma_i \frac{\left[c_i^{ext} + \sum_{j \in FO(i)} \alpha_j x_j \right]}{x_i}$$

$$T x_i = \beta_i x_i + \gamma_i \left(c_i^{ext} + \sum_{j \in FO(i)} \alpha_j x_j \right)$$

$$Tx = \operatorname{diag}(\beta)\, x + \operatorname{diag}(\gamma)\, c^{ext} + \underbrace{\operatorname{diag}(\gamma)\, F \operatorname{diag}(\alpha)}\, x$$

$$\underbrace{\begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{bmatrix}} \qquad \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix} \qquad \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix}$$

$$K = \begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{bmatrix} + \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix} F \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix}$$

$$Tx = Kx + \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix} c^{ext}$$

$$Tx - Kx = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix} c^{ext} \qquad T > \max_i \rho_i$$

$$(TI - K)x = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix} c^{ext}$$

$$x = \boxed{(TI - K)^{-1}} \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix} c^{ext}$$

$$\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix}$$

$$(T - \rho_i)$$