Controllability and State Transfer

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Controllability and State Transfer

- Infinite Horizon min norm state transfer
- State transfer
- Reachable set
- Controllability Matrix
- Minimum Norm inputs

State Transfer (1.3)

$\dot{x} = Ax + Bu \qquad x_{k+1} = \Phi x_k + \Gamma u_k \qquad (t_i, t_f)$

$u: (t_i, t_f) \to \mathbb{R}^m$ steers or transfers $x(t_i)$ to $x(t_f)$.

- when can $x(t_f)$ be given $x(t_i)$ @ $t = t_f$.
- how quickly can I get x to x_{target}.
- how do I determine u that gets me there
- how do I find a "small" u or "efficient" that gets me from $x(t_i)$ to $\underline{x(t_f)}$ — x_{des}

State Transfer (2.3)

$t_i = \phi$. $x(t_i) = x_0 = \phi$. relaxed state

$x(t)$ is <u>REACHABLE</u> if in t seconds

$R_t \subseteq \mathbb{R}^n$ set of points $x(t)$ can be in t seconds.

$$R_t = \left\{ \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \,\middle|\, u[0,\tau] \to \mathbb{R}^m \right\}$$

$$\dot{x} = Ax + Bu$$

State Transfer (3.3)

$x_{k+1} = \Phi x_k + \Gamma u_k$

$$R_t = \left\{ \sum_{\tau=0}^{t-1} \Phi^{t-1+\tau} \Gamma u(\tau) \,\middle|\, u_k \in \mathbb{R}^m, \; k=0 \ldots t-1 \right\}$$

R_t subspace of $\mathbb{R}^n$

$R_t \subseteq R_s$ if $t \leq s$ — can always reach more points given more time

Reachability (1.3)

Reachability (2.3)

Reachability (3.3)

Reachability for Discrete-Time LDS (1.2)

$x_{k+1} = \Phi x_k + \Gamma u_k \qquad x_k \in \mathbb{R}^n$

$x(0) = 0$

$x_1 = \Gamma u_0$

$x_2 = \Phi \Gamma u_0 + \Gamma u_1$

$x_3 = \Phi^2 \Gamma u_0 + \Phi \Gamma u_1 + \Gamma u_2$

$\vdots$

$$x_k = \mathcal{C}_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$\mathcal{C}_k = [\Gamma \;\; \Phi\Gamma \;\; \Phi^2\Gamma \; \cdots \; \Phi^{k-1}\Gamma]$ ← controllability matrix
Krylov matrix

$R_k = \text{range}(\mathcal{C}_k)$

Reachability for Discrete-Time LDS (2.2)

C-H → Φ^k is a linear combination of $\Phi^0 \cdots \Phi^{n-1}$

$k \geq n \qquad R_k \triangleq \text{Range}(\mathcal{C}_k) = \text{Range}(\mathcal{C}_n)$.

$$R_k = \begin{cases} \text{range}(\mathcal{C}_k) & k < n \\ \text{Range}(\mathcal{C}_n) & k \geq n \end{cases}$$

$\mathcal{C} \triangleq [B \;\; AB \;\; A^2B \; \cdots \; A^{n-1}B]$ range($\mathcal{C}$) anything I can hit with x.

if ($\mathcal{C}$) is full rank → $R_t \to \mathbb{R}^n$.

Controllable System (1.2)

Controllable: if all states are reachable in n steps.

$\mathcal{R}_t = \mathbb{R}^n \quad \text{Rank}(\mathcal{C}) = n$.

Controllable System (2.2)

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k$$

$$\Phi r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{rank}(\mathcal{C}) \neq 2$$

$$R = \text{range}(\mathcal{C}) = \{ x \mid x_1 = x_2 \}$$

General State Transfer (1.2)

$t_f > t_i$

$$x(t_f) = \Phi^{t_f - t_i} x(t_i) + \mathcal{C}_{t_f - t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}$$

$x(t_i) \longrightarrow x_{des}$ @ $t = t_f$.

- General State Transfer $\longleftrightarrow$ Reachability Problem
- If CONTROLLABLE, I can get there in n steps
- Drive $x \rightarrow \{0\}$ (regulator)
- If state x starts in R, will never leave R.

General State Transfer (2.2)

$z \in \text{range}(\mathcal{C}) \rightarrow \Phi z \in \text{Range}(\mathcal{C})$

$$z = \sum_{i=1}^{n-1} \Phi^i \Gamma u_i$$

$$\Phi z = \sum_{i=0}^{n-1} \Phi^{i+1} \Gamma u_i = \sum_{i=1}^{n-1} \Phi^i \Gamma u(i-1) + \Phi^n \Gamma u(n-1)$$

$$= \sum_{i=0}^{n-1} \Phi^i \Gamma \left(u(i-1) - \underline{\alpha_i} u(n-1)\right) - \underline{\alpha_0} u(n-1)\Gamma \in \text{Range}(\mathcal{C})$$

$$\mathcal{X}(s) = z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_0 = \det(zI - \Phi)$$

$$\Phi^n = -\alpha_0 I - \dots \alpha_{n-1} \Phi^{n-1}$$

Least-Norm Input for Reachability (1.4)

$\text{Rank}(\mathcal{C}) = n \quad x(0) = 0 \text{ to } x_k = x_{des}$

$$x_{des} = \mathcal{C}\begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

FAT, FULL RANK ($\mathcal{C}$); U (the input vector)

$\min \|U\|^2$

subj to: $x_{k+1} = \Phi x_k + \Gamma u_k$

$x_k = x_{des}$.

$$\begin{bmatrix} u_{ln}(k-1) \\ \vdots \\ u_{ln}(0) \end{bmatrix} = \mathcal{C}_k^T \left(\mathcal{C}_k \mathcal{C}_k^T\right)^{-1} x_{des} \leftarrow \text{LEAST NORM}$$

Least-Norm Input for Reachability (2.4)

$$u_{ln}(\tau) = \Gamma^T (\Phi^T)^{t-1-\tau} \left[\sum_{s=0}^{t-1} \Phi^s \Gamma \Gamma^T (\Phi^T)^s \right]^{-1} x_{des}$$

$\tau = 0 \ldots t-1$ "MATCHED FILTERING"

Least-Norm Input for Reachability (3.4)

$\mathcal{E}_{min}$ min value of $\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$ required to reach x_{des} in τ epochs

$$\mathcal{E}_{min} = \sum_{\tau=0}^{t-1} \|u_{ln}(\tau)\|^2 = \left(\mathcal{C}_k^T(\mathcal{C}_k\mathcal{C}_k^T)^{-1}x_{des}\right)^T \mathcal{C}_k^T(\mathcal{C}_k\mathcal{C}_k^T)^{-1}x_{des}$$

$$= x_{des}^T(\mathcal{C}_k\mathcal{C}_k^T)^{-1}\underbrace{\mathcal{C}_k\mathcal{C}_k^T(\mathcal{C}_k\mathcal{C}_k^T)^{-1}}_{I}x_{des}$$

$$= x_{des}^T(\mathcal{C}_k\mathcal{C}_k^T)^{-1}x_{des}$$

Least-Norm Input for Reachability (4.4)

$$\mathcal{E}_{min} = x_{des}^T \left(\sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1} x_{des}$$

$\mathcal{E}_{min}(x_{des}, t)$ measure of how hard it is to reach x_{des} in t epochs from $x(0) = 0$.

$\mathcal{E}_{min}(x_{des}, t)$ provides a measure of controllability/reachability

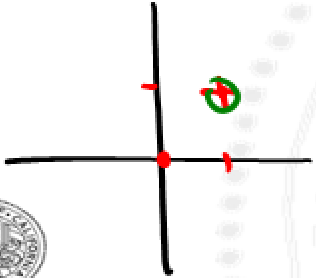
Ellipsoid $\{ z \mid \mathcal{E}_{min}(z, t) \le 1 \}$ reachable in 1 energy

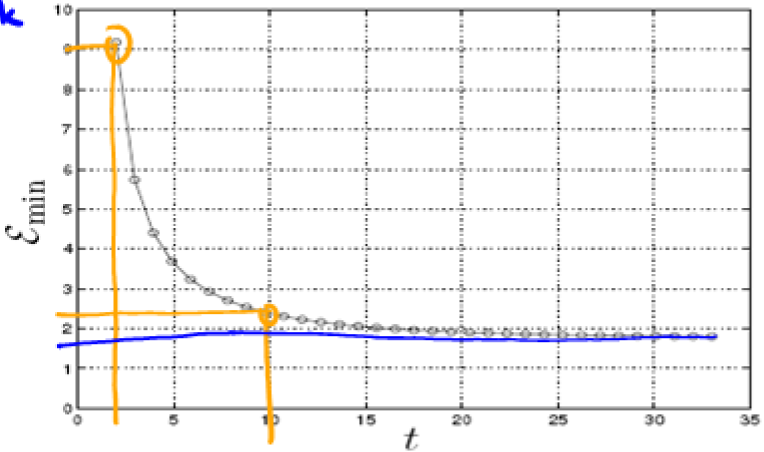
$\mathcal{E}_{min} \simeq f(t)$ $\quad t \ge s \quad \mathcal{E}_{min}(t) \le \mathcal{E}_{min}(s)$.

Least-Norm Input Example (1.2)

$$x_{k+1} = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 6 \end{bmatrix} u_k$$

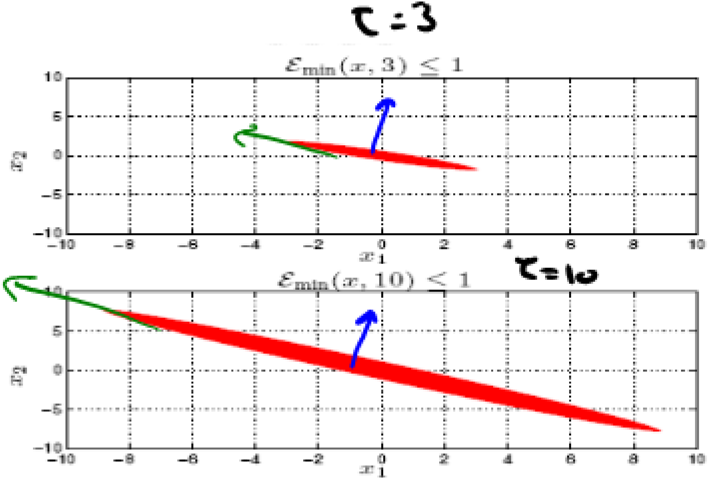
$$\mathcal{E}_{min}(z, t) \qquad z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





Least-Norm Input Example (1.2)

$\mathcal{E}_{min}(z, 3) \leq 1$



Minimum Energy of Infinite Horizon (1.3)

$$P = \lim_{t \to \infty} \left(C_t C_t^T \right)^{-1} = \lim_{t \to \infty} \left(\sum_{\tau=0}^{t-1} \Phi^\tau \Gamma \Gamma^T (\Phi^T)^\tau \right)^{-1}$$

min energy to get to x_{des} for any amount of time

$$\min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \,\middle|\, x(0)=0,\ x(t)=x_{des} \right\} = x_{des}^T P x_{des}$$

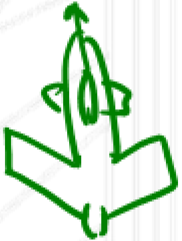
if Φ is <u>STABLE</u> $P > 0$ "can't get anywhere for free"

Minimum Energy of Infinite Horizon (2.3)

if ϕ is **NOT** stable P can have a non-zero nullspace.

$P_z = \phi \rightarrow z \neq 0$. I can get to z using u's that are very small control energy.

Basis of highly maneuverable military aircraft



X-29 28%.

Minimum Energy of Infinite Horizon (3.3)

Continuous-time Reachability (1.3)

$$\dot{x} = Ax + Bu \qquad x(t) \in \mathbb{R}^n$$

$$R_t = \left\{ \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau \;\middle|\; u[0:t] \to \mathbb{R}^m \right\}$$

for $t > 0$ $\quad R_t = R = \text{Range}(\mathcal{C})$

$$\mathcal{C} \triangleq \left[B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B \right]$$

Same condition as the discrete system.

Continuous-time Reachability (2.3)

Any reachable point can be reached in an infinitesimal amount of time given large enough u.

for any u, $x(0) = 0$. $x(t) \in \text{Range}(\mathcal{C})$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$ use C-H for $A^n, A^{n+1}, \dots$

$$x(t) = \int_0^t e^{At} B u(\tau)\, d\tau$$

Continuous-time Reachability (3.3)

$$x(t) = \int_0^t \left(\sum_{i=0}^{n-1} \alpha_i(\tau) A^i \right) B u(t-\tau)\, d\tau$$

$$= \underbrace{\sum_{i=0}^{n-1} A^i B}_{\mathcal{C}} \underbrace{\int_0^t \alpha_i(\tau) u(t-\tau)\, d\tau}_{z} = \mathcal{C} z$$

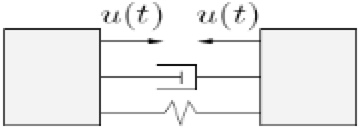
$$z_i = \int_0^t \alpha_i(\tau) u(t-\tau)\, d\tau$$

Impulsive Inputs (1.3)

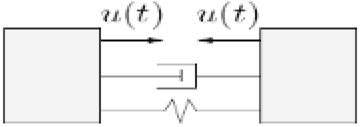
Impulsive Inputs (2.3)

Impulsive Inputs (3.3)

Example (1.2)



Example (2.2)



Least-norm input for Reachability (1.3)

Least-norm input for Reachability (2.3)

Least-norm input for Reachability (3.3)

Minimum Energy of Infinite Horizon (1.3)

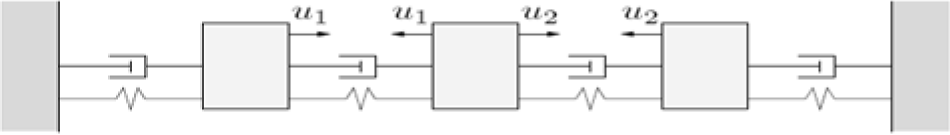
Minimum Energy of Infinite Horizon (2.3)

Minimum Energy of Infinite Horizon (3.3)

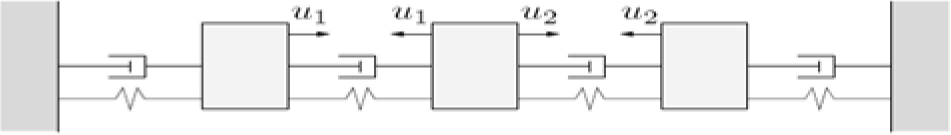
General State Transfer (1.2)

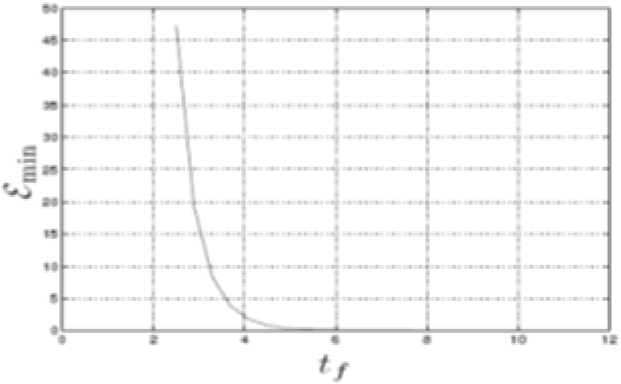
General State Transfer (2.2)

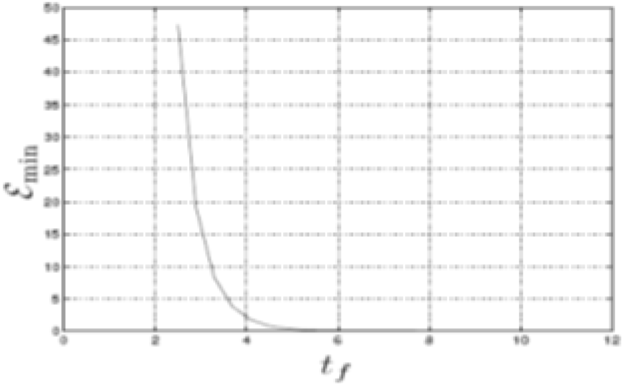
Example (1.4)



Example (2.4)


u_1
u_1
u_2
u_2


$\mathcal{E}_{\min}$
t_f
0
5
10
15
20
25
30
35
40
45
50
0
2
4
6
8
10
12


$\mathcal{E}_{\min}$
t_f
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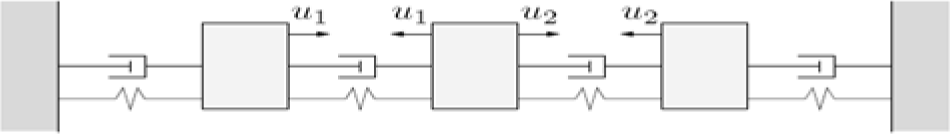
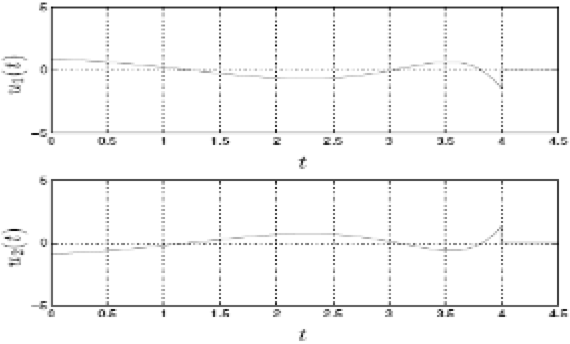
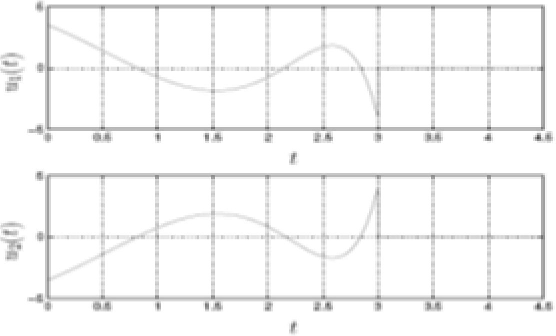
12

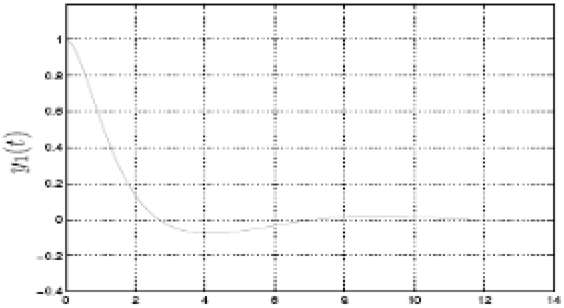
Example (3.4)







Example (4.4)


$y_1(t)$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
2
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6
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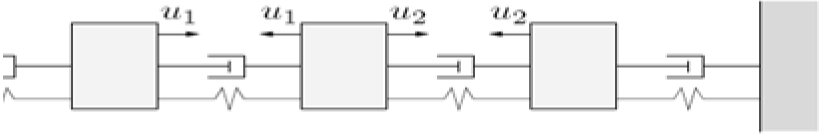
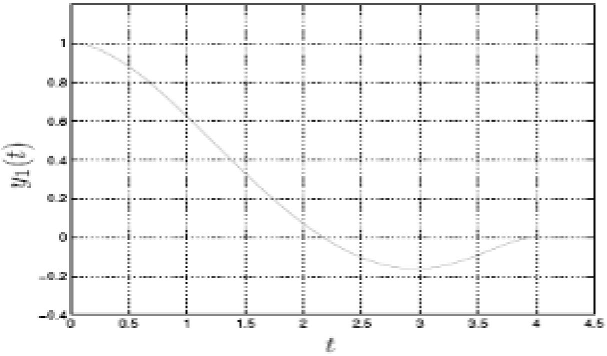
10
12
14


u_1
u_1
u_2
u_2


$y_1(t)$
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0.8
0.6
0.4
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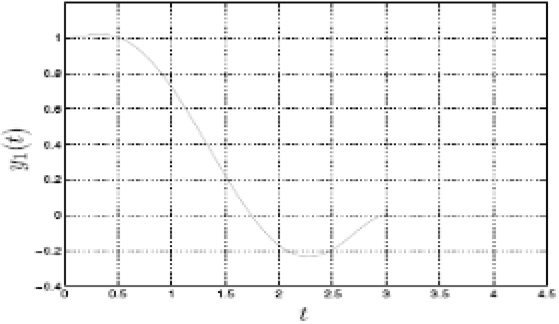
-0.2
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
t


$y_1(t)$
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0.8
0.6
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Questions?