Symmetric Matrices, Quadratic Forms, Matrix Norm, and SVD

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Symmetric Matrices, Quadratic Forms, Matrix Norm, and the SVD

- Singular Value Decomposition
- Eigenvectors of symmetric matrices
- Quadratic Forms
- Inequalities for quadratic forms
- Positive semi-definite matrices
- Norm of a matrix

Eigenvalues of Symmetric Matrices (1.2)

$A \in \mathbb{R}^{n \times n}$ $\quad A = A^T \quad$ symmetric

fact: eigenvalues of A are all real

$$Av = \lambda v \quad v \neq 0 \quad v \in \mathbb{C}^n$$

$$\bar{v}^T A v = \bar{v}^T(\lambda v) = \lambda \bar{v}^T v = \lambda \sum_{i=1}^{n} |v_i|^2$$

$$\bar{v}^T A v = (\overline{Av})^T v = (\overline{\lambda v})^T v = \bar{\lambda} \bar{v}^T v = \bar{\lambda} \sum_{i=1}^{n} |v_i|^2$$

$$\lambda = \bar{\lambda} \quad \text{ie:} \quad \lambda \in \mathbb{R} \quad v \in \mathbb{R}^n$$

Eigenvalues of Symmetric Matrices (2.2)

$\dot{x} = Ax$ $\quad A = A^T$ $\quad$ no oscillations $\quad$ "self-adjoint"

choose an orthogonal set of eigenvectors of A $(=A^T)$

$q_1 \cdots q_n$ such that $Aq_i = \lambda q_i$ and $q_i^T q_j = \delta_{ij}$

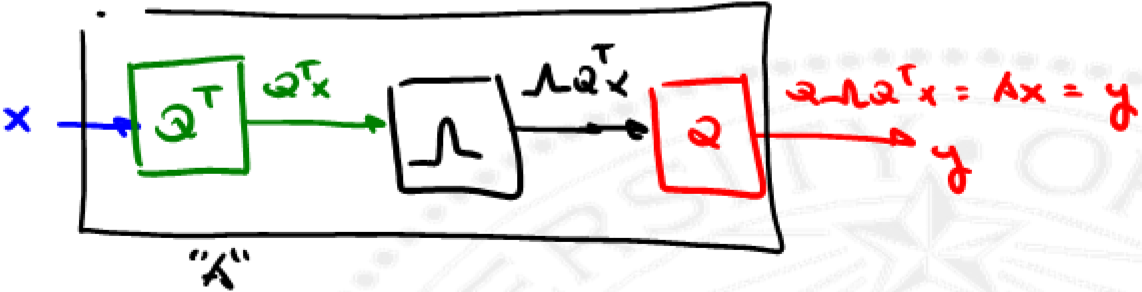
$[q_1 \cdots q_n] \triangleq Q$ orthogonal matrix $Q^T = Q^{-1}$

$$Q^{-1} A Q = Q^T A Q = \Lambda$$

left/right eigenvectors are the same.

$$A = Q \Lambda Q^T = \sum_{i=1}^{n} \lambda_i \underline{q_i q_i^T}$$ ← rank 1 dyadic expansion

Eigenvectors of Symmetric Matrices (1.2)



- Resolve x into q_i coordinates
- Scale those coordinates by λ_i } dilation
- Reconstruct y with basis of Q.

Eigenvectors of Symmetric Matrices (2.2)

$$A = \sum_{i=1}^{n} \lambda_i \underline{q_i q_i^T}$$

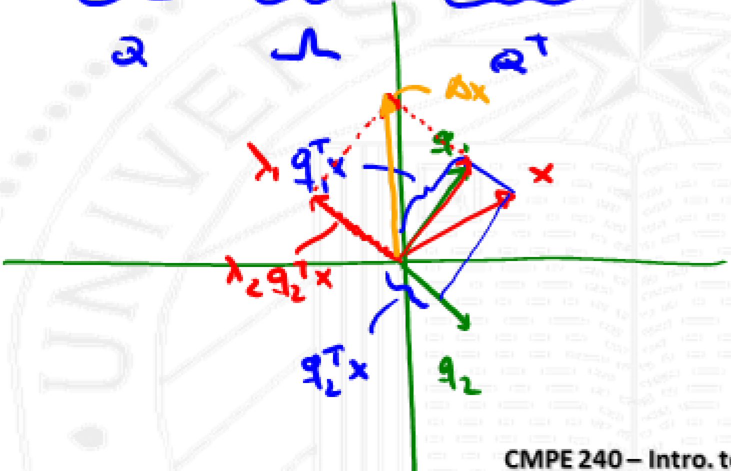
n×n rank 1 dimd

orthogonal projection spanned by

the line q_i

Interpretation (1.3)

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \Bigg(\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}}_{\Lambda} \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)}_{Q^T} \Bigg)$$



Interpretation (2.3)

λ_i are distinct

suppose $v_1 \cdots v_n$ are a set of linearly independent eigenvectors of A.

$\|v_i\| = 1$

$Av_i = \lambda_i v_i$

$\uparrow$

distinct

$v_i^T(Av_j) = (Av_i)^T v_j = \lambda_i v_i^T v_j = \lambda_j v_i^T v_j$

$(\lambda_i - \lambda_j) v_i^T v_j = 0 \quad v_i^T v_j = 0 \quad v_i \perp v_j$

Interpretation (3.3)

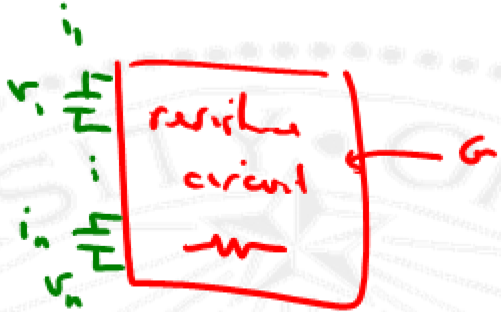
Example (1.5)

$c_k \dot{v}_k = -i_k$

$i = Gv$

$G \triangleq$ conductance matrix

$G = G^T \in \mathbb{R}^{n \times n}$



Example (2.5)

$$\dot{v} = -\overbrace{C^{-1} G}^{\text{not necessarily symmetric}} v$$

$$x_i \triangleq \sqrt{c_i}\, v_i$$

$$x = C^{1/2} v \rightarrow v = C^{-1/2} x$$

$$C = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

$$\dot{x} = C^{1/2} \dot{v}$$

$$C^{1/2} = \begin{bmatrix} \sqrt{c_1} & & \\ & \ddots & \\ & & \sqrt{c_n} \end{bmatrix}$$

$$\dot{x} = -C^{1/2} C^{-1} G C^{-1/2} x$$

$$\dot{x} = -\underbrace{C^{-1/2} G C^{-1/2}}_{\text{symmetric}} x$$

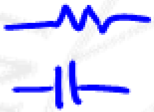
$\approx \frac{1}{2}\|x\|^2$ energy

Example (3.5)

$C^{-1/2} G C^{-1/2}$ – symmetric, all eigenvalues are real

so are the eigenvalues of $-C^{-1}G$

choose eigenvectors to be orthogonal



cannot build an oscillator

Example (4.5)

$s_i = C^{-1/2} q_i$ satisfies

$-C^{-1} G s_i = \lambda s_i \longrightarrow s_i \perp s_j \quad s_i^T s_j = \delta_{ij}$

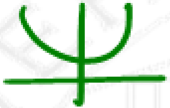
Example (5.5)

Quadratic Forms (1.3)

$f: \mathbb{R}^n \to \mathbb{R}$ has the form

$$f(x) = x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

quadratic form



can assume that $A = A^T$

$$x^T A x = x^T \underbrace{\left(\frac{A + A^T}{2}\right)}_{\text{symmetric part of } A.} x$$

Quadratic Forms (2.3)

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad x^T A x = (x_1, x_2)\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2)\begin{bmatrix} x_1 + 3x_2 \\ 2x_2 \end{bmatrix}$$

$$= \underline{x_1^2 + 3x_1 x_2 + 2x_2^2}$$

$$\frac{A + A^T}{2} = \frac{\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}}{2} = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix} \qquad x^T\left(\frac{A + A^T}{2}\right)x = (x_1, x_2)\begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x^T\left(\frac{A + A^T}{2}\right)x = (x_1, x_2)\begin{bmatrix} x_1 + 3/2\, x_2 \\ 3/2\, x_1 + 2x_2 \end{bmatrix} = x_1^2 + \frac{3}{2}x_1 x_2 + \frac{3}{2}x_1 x_2 + 2x_2^2$$

$$= \underline{x_1^2 + 3x_1 x_2 + 2x_2^2}$$

skew symmetric $S^T = -S$

$$[\omega \times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

cross product matrix

$$x^T S x = x^T(-S^T)x \longrightarrow \phi .$$

Quadratic Forms (3.3)

$$\|Bx\|^2 = (Bx)^T(Bx) = x^T \underline{B^TB}\, x$$

already symmetric

$$\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \longrightarrow x^T \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} x$$

measure of how bumpy x is

$$\|Fx\|^2 - \|Gx\|^2 \longrightarrow x^T(F^TF - G^TG)\,x$$

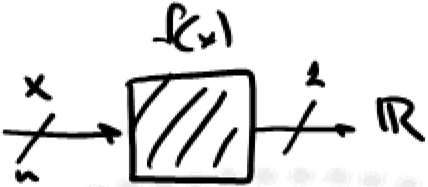
$\{x \mid x^T A x = a\} \longrightarrow$ quadratic surface in $\mathbb{R}^n$

$\{x \mid x^T A x \leq a\} \longrightarrow$ quadratic region/volume in $\mathbb{R}^n$

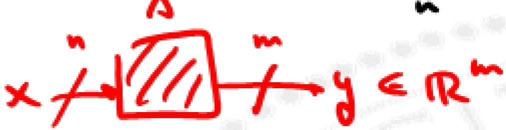
$\{x \mid x^T x = 1\} \longleftarrow$ unit hyper sphere in $\mathbb{R}^n$

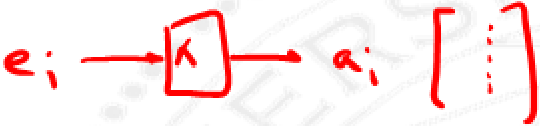
$\{x \mid x^T x \leq 1\} \leftarrow$ unit <u>ball</u>.

$$f(x) \doteq x^T A x$$



$$y = Ax$$





$$f(e_i) = e_i^T A e_i = A_{ii}$$

$$f(e_i + e_j) = (e_i + e_j)^T A (e_i + e_j) = A_{ii} + 2A_{ij} + A_{jj}$$

Inequalities for Quadratic Forms (1.3)

$A = A^T \in \mathbb{R}^{n \times n}$ $\qquad A = Q \Lambda Q^T$

by just $\rightarrow \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \leftarrow$ smallest

$x^T A x = x^T Q \Lambda Q^T x = (Q^T x)^T \Lambda (Q^T x)$

$y = Q^T x$ $\qquad = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$

$= \sum_{i=1}^{n} \lambda_i (q_i^T x)^2$ $\qquad \|Q^T x\|^2 = \|x\|^2$

$\lambda_1 \|x\|^2 \geq x^T A x \geq \lambda_n \|x\|^2$

Inequalities for Quadratic Forms (2.3)

$$\boxed{\lambda_{max} \|x\|^2 \geq x^T \Lambda x \geq \lambda_{min} \|x\|^2}$$

$$q_1^T \Lambda q_1 = \lambda_{max} \|q_1\|^2 = \lambda_{max}$$

$$q_n^T \Lambda q_n = \lambda_{min} \|q_n\|^2 = \lambda_{min}$$

Gain in quadratic form

$$\frac{x^T \Lambda x}{\|x\|^2} \triangleq \text{Raleigh Quotient}$$

$$\lambda_{max} \geq \frac{x^T \Lambda x}{x^T x} \geq \lambda_{min}$$

Inequalities for Quadratic Forms (3.3)

$A = A^T \in \mathbb{R}^{n \times n}$

semi

A is positive definite if $x^T A x \geq 0 \; \forall \; x$

$\boxed{A \geq 0}$ $\min \operatorname{eig}\left(\frac{A^T + A}{2}\right) \geq 0$

$A \geq 0$ if and only if $\lambda_{min}(A) \geq 0$

Positive semi-definite and Positive definite matrices (1.3)

$A \leq 0$ negative semi-definite $\quad \underline{-A \geq 0.}$

$A < 0$ negative definite $\quad \underline{-A > 0}$

<u>A is indefinite</u>

Positive semi-definite and Positive definite matrices (2.3)

if $B = B^T \in \mathbb{R}^{n \times n}$

$A \geq B$ if $\underline{(A-B) \geq 0}$. positive semi-definite

$A < B \longrightarrow (A-B) < 0$ or $(B-A) > 0$

$A > B \longrightarrow x^T A x > x^T B x \quad \forall x \neq 0$.

Positive semi-definite and Positive definite matrices (3.3)

$A \geq B \quad C \geq D \longrightarrow A + C \geq B + D$

$B \leq 0 \longrightarrow A + B \leq A$

$A \geq 0, \ \alpha \geq 0 \longrightarrow \alpha A \geq 0.$

$A \geq 0 \longrightarrow A^2 \geq 0$

$A > 0 \longrightarrow A^{-1} > 0$

$A \geq B \;\cancel{\longrightarrow}\; B \geq A$

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \overset{?}{\leq} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

"incomparable"

Matrix Inequalities (1.3)

$$x^T A x \le 1 \longrightarrow x^T A^{-1} x \le 1$$

Matrix Inequalities (2.3)

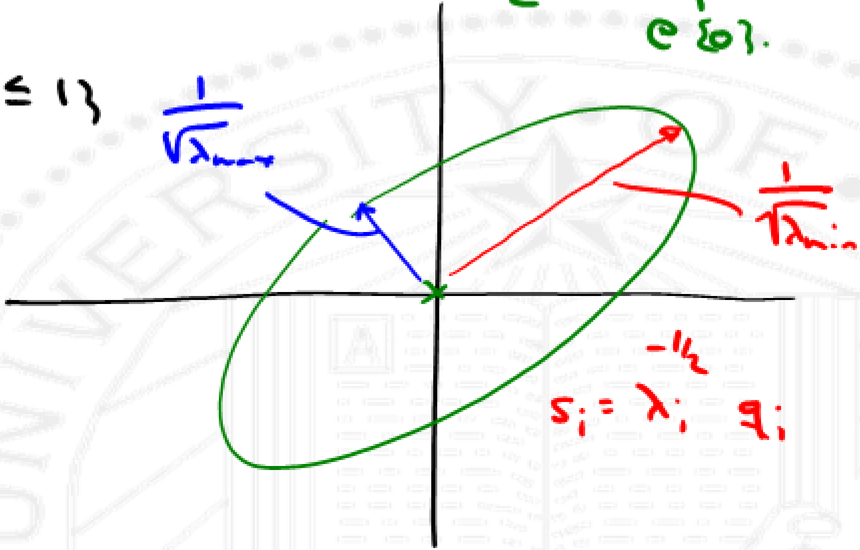
Matrix Inequalities (3.3)

Ellipsoids (1.3)

$A = A^T > 0$

$\mathcal{E} \triangleq \{x \mid x^T A x \leq 1\}$

$\mathcal{E} \triangleq$ ellipsoid centered @ $\{0\}$.



Ellipsoids (2.3)

eigenvalues – length of semi-major axes

eigenvectors – direction of the semi-major axes

$s_i = \lambda_i^{-1/2} q_i$

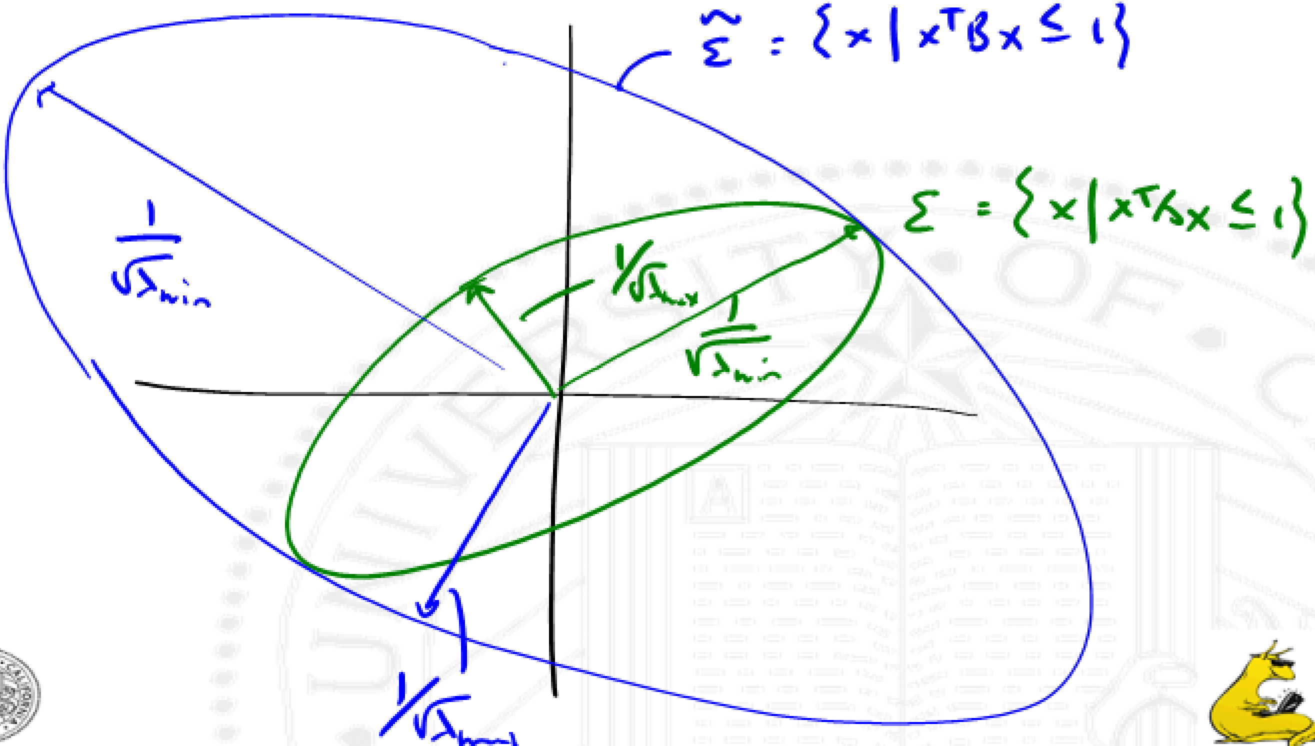
q_1 $x^T A x$ is large ellipsoid is THIN in that direction

q_n $x^T A x$ is small ellipsoid is FAT in that direction

Ellipsoids (3.3)

$\sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \triangleq$ maximum eccentricity of A
"condition number"

$\hat{\mathcal{E}} = \{ x \mid x^T B x \leq 1 \}$ $B > 0$ $\mathcal{E} \subseteq \hat{\mathcal{E}}$ $A \geq B$



$\tilde{\Sigma} : \{x \mid x^T B x \leq 1\}$
$\Sigma : \{x \mid x^T A x \leq 1\}$
$\frac{1}{\sqrt{\lambda_{min}}}$
$1/\sqrt{\lambda_{max}}$
$\frac{1}{\sqrt{\lambda_{min}}}$
$1/\sqrt{\lambda_{max}}$

Gain of a Matrix in a Direction (1.3)

$A \in \mathbb{R}^{m \times n}$ not necessarily square or symmetric

$y = Ax \quad x \in \mathbb{R}^n$ $\dfrac{\|Ax\|}{\|x\|}$ gives an amplification factor.

gain of the matrix A in the direction x.

Gain of a Matrix in a Direction (2.3)

$\max \dfrac{\|Ax\|}{\|x\|} \quad x \neq 0.$

Spectral norm of A.

l_2 norm of A

$\|A\|_2$

$\lambda_{max}(A^T A)$

Gain of a Matrix in a Direction (3.3)

$$\max_{x} \frac{\|Ax\|}{\|x\|} = \lambda_{max}(A^T A)$$

$A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^n$

suj: $x \neq 0$

Norm of a matrix

$$\sqrt{\lambda_{max}(A^TA)} \geq \|A\| \geq \sqrt{\lambda_{min}(A^TA)}$$

$A^TA \in \mathbb{R}^{n \times n}$ is symmetric and $A^TA \geq 0$.

so that $\lambda_{max}, \lambda_{min} \geq 0$.

max gain $x = q_1$ eig (A^TA) with λ_1

min gain $x = q_n$ eig (A^TA) with λ_n

Matrix Norm (1.4)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$e_1 \longrightarrow x^T A x \sim 6$

$e_2 \longrightarrow x^T A x \sim 7$

$A^T A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.62 & 0.785 \\ 0.785 & -0.62 \end{bmatrix} \begin{bmatrix} 90.7 & 0 \\ 0 & 0.265 \end{bmatrix} \begin{bmatrix} Q^T \end{bmatrix}$

2 Λ

$\|A\| = \sqrt{\lambda_{max}(A^T A)} = \underline{9.53}$

$x = \begin{bmatrix} 0.62 \\ 0.785 \end{bmatrix} \rightarrow \|x\| = 1 \quad \|Ax\| \simeq \underline{9.53}$

$x = \begin{bmatrix} 0.785 \\ -0.62 \end{bmatrix} \rightarrow \|x\| = 1 \rightarrow \|Ax\| \simeq 0.5$

Matrix Norm (2.4)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$0.54 \le \|A\| \le 9.53$$

$$x = \begin{bmatrix} 0.785 \\ -0.62 \end{bmatrix} \qquad \begin{bmatrix} 0.62 \\ 0.785 \end{bmatrix}$$

Matrix Norm (3.4)

consistent with vector norm

$a \in \mathbb{R}^{n \times 1} \longrightarrow \sqrt{\lambda_{max}(a^T a)} = \sqrt{a^T a} = \|a\|$.

for any x $\quad \|Ax\| \leq \|A\| \|x\|$. $\leftarrow$ along direction of $\operatorname{eig}(A^T A)$.

Matrix Norm (4.4)

$$\|aA\| = |a|\,\|A\|$$

$$\|A+B\| \le \|A\| + \|B\|$$

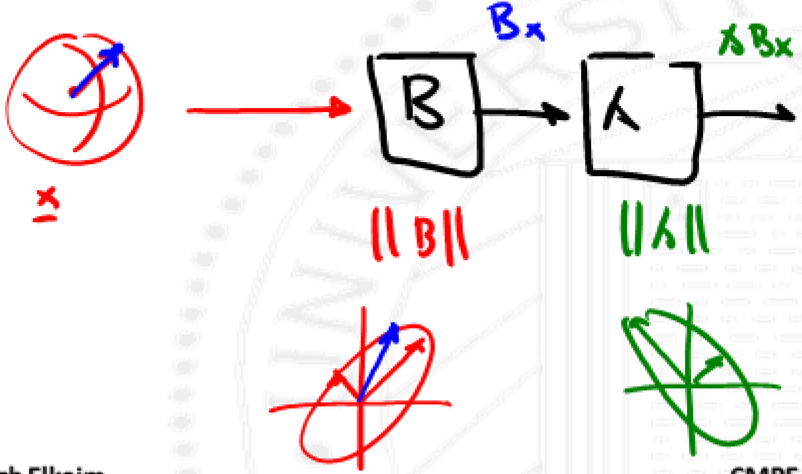
$$\|A\| = 0 \longrightarrow A = 0$$

$$\boxed{\|AB\| \le \|A\| \cdot \|B\|}$$

Properties of Matrix Norm (1.3)

geometric interpretation

$$\|AB\| \le \|A\| \cdot \|B\|$$



$$\frac{x^T (AB)^T (AB)\, x}{x^T x}$$

Properties of Matrix Norm (2.3)

$A \in \mathbb{R}^{n \times n} \longrightarrow A = T \Lambda T^{-1}$ eigvalues/eigvectors

$A = A^T \in \mathbb{R}^{n \times n} \longrightarrow A = Q \Lambda Q^T$, symmetric eigvectors

$A \in \mathbb{R}^{m \times n} \longrightarrow$ SVD: $A = U \Sigma V^T$

Properties of Matrix Norm (3.3)

Singular Value Decomposition (1.3)

SVD of $A \in \mathbb{R}^{m \times n}$ $\qquad A = U \Sigma V^T$

$A \in \mathbb{R}^{m \times n}$ $\quad \text{rank}(A) = r$

$U \in \mathbb{R}^{m \times r}$ $\quad U^T U = I_{r \times r}$

$V \in \mathbb{R}^{n \times r}$ $\quad V^T V = I_{r \times r}$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$

SINGULAR VALUES of A

Singular Value Decomposition (2.3)

$$\underbrace{A}_{m \times n} = \underbrace{U}_{m \times r}\ \underbrace{\Sigma}_{r \times r}\ \underbrace{V^T}_{r \times n}$$

$U = [u_1 \cdots u_r]$ $\quad u_i \in \mathbb{R}^m$

$V = [v_1 \cdots v_r]$ $\quad v_i \in \mathbb{R}^n$

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \underbrace{u_i v_i^T}_{\text{rank } 1} \sim \lambda_i \overbrace{v_i w_i^T}$$

max diadic form

Singular Value Decomposition (3.3)

$\sigma_i \triangleq$ singular value of A.

$v_i \triangleq$ right (input) singular vector of A

$u_i \triangleq$ left (output) singular vector of A

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma \underbrace{U^TU}\Sigma V^T = \underline{V\Sigma^2V^T}$$

$v_i \triangleq$ eigenvectors of A^TA corresponding to non-zero λ's.

$\sigma_i \triangleq \sqrt{\lambda_i(A^TA)} \qquad \lambda_i(A^TA) = 0 \ \forall\ i > r.$

Interpretations (1.3)

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V \Sigma U^T = \underline{U\Sigma^2 U^T}$$

$u_i \triangleq$ eigenvectors of AA^T associated with non-zero λ's.

$$\operatorname{eig}(AB) = \operatorname{eig}(BA)$$

$$\sigma_i = \sqrt{\lambda_i(AA^T)} \quad \leftrightarrow \quad \lambda_i(AA^T) = 0 \ \forall\, i > r.$$

Interpretations (2.3)

u_i – eigenvectors of (AA^T) LEFT/OUTPUT direction

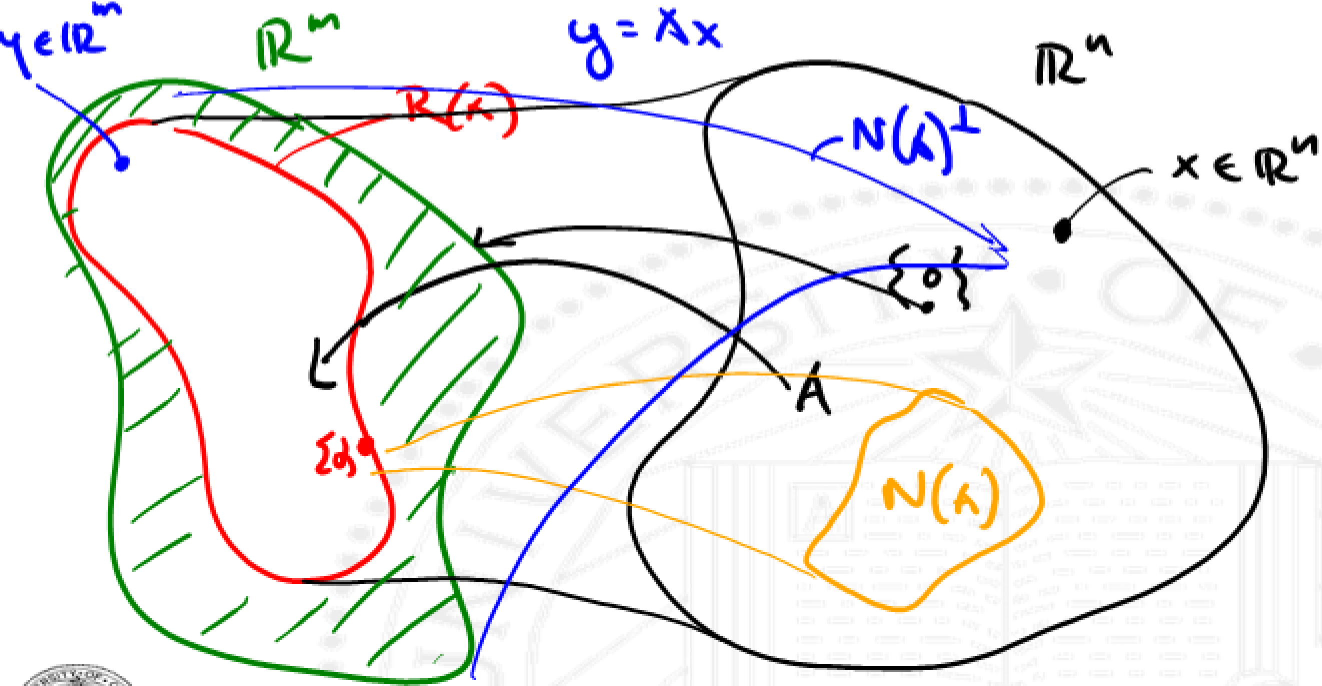
v_i – eigenvectors of (A^TA) RIGHT/INPUT direction

σ_i – eigenvalues of (A^TA) or (AA^T) up to rank of A.

$$\sqrt{\lambda_i(A^TA)} = \sqrt{\lambda_i(AA^T)}$$

$[u_1 \cdots u_r]$ – orthonormal basis for $R(A)$

$[v_1 \cdots v_r]$ – orthonormal basis for $N(A)^\perp$



$y \in \mathbb{R}^m$
$\mathbb{R}^m$
$y = Ax$
$\mathbb{R}^n$
$R(A)$
$N(A)^\perp$
$x \in \mathbb{R}^n$
$\{0\}$
A
$\{y\}$
$N(A)$

Interpretations (3.3)

$R(U) = R(A)$

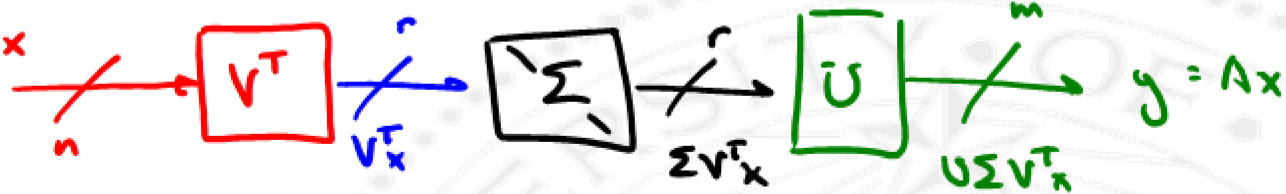
$N(A)^{\perp} = R(V)$

Any linear mapping

$y = Ax$

$y \in \mathbb{R}^m \quad x \in \mathbb{R}^n$

$$A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad y = Ax$$



resolve x into v input directions

Dilate along coordinate axes

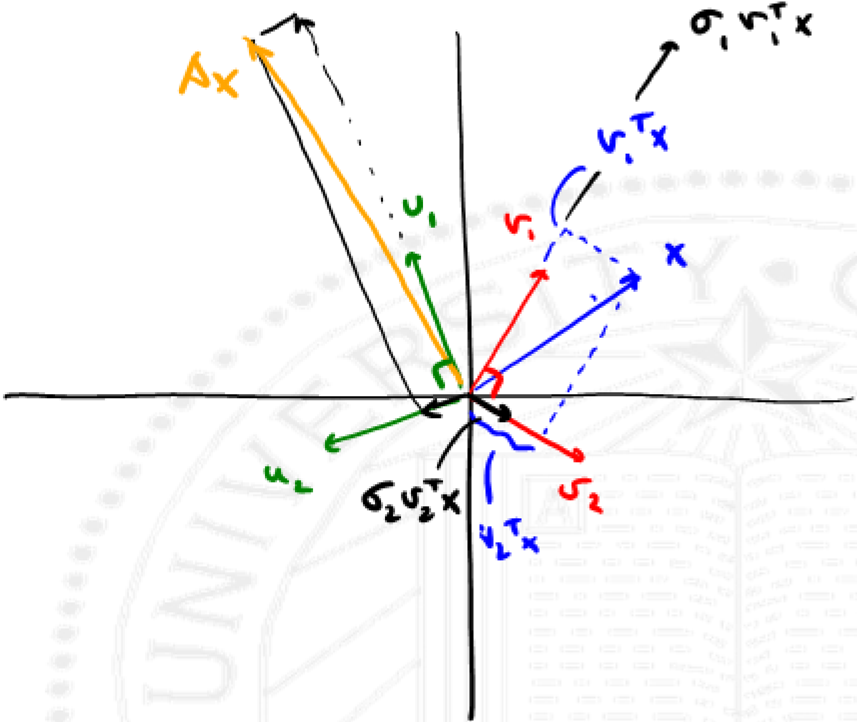
reshape the results along the output directions

(1) Compute coefficient of $\underline{x}$ along each input direction

(2) Scale that coefficient by σ_i

(3) Reconstruct y along the output direction

$A \in \mathbb{R}^{2\times 2}$



Questions?