$$\dot{x} = Ax + Bu$$

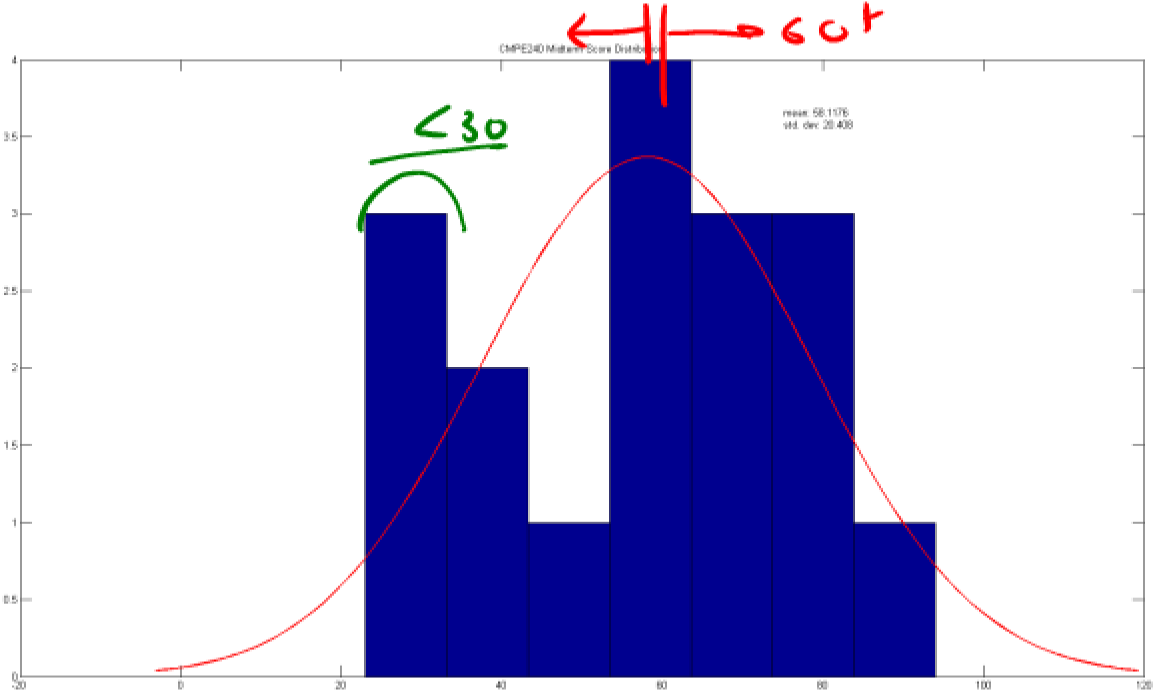
input

output

$$y = Cx + Du$$

Linear Dynamical Systems with Inputs and Outputs

Gabriel Hugh Elkaim



CMPE240 Midterm Score Distribution
mean: 58.1176
std. dev: 20.408
60+
<30
-20 0 20 40 60 80 100 120
0 0.5 1 1.5 2 2.5 3 3.5 4

$A \in \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{n \times n}$ is itself a vector space (n^2)

vec(A) $A(:)$, reshape(A)...

$A \in \mathbb{R}^{n \times n}$ $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ position $E_{ij} = e_i e_j^T$ $i,j = 1..n$

rank 1 entry

$A \in \mathbb{R}^{10 \times 10}$ $\underline{A_1, A_2, \cdots A_{101}} \in \mathbb{R}^{10 \times 10}$ $A = \sum_{i,j}^{n} \alpha E_{ij}$

are **NOT** independent $I, A_1 \ldots A_{101}$

$p(s) \longleftrightarrow p(A)$ $\quad A^k - I, A, \ldots, A^{n-1}$ $\quad A \in \mathbb{R}^{n \times n}$

$$e^A \triangleq I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

$$A \in \mathbb{R}^{3 \times 3} \longrightarrow e^A = p(I, A, A^2)$$

ZIPFEL

R ← Rotation Matrix (DCM)

$\omega : \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ rotation rate

$$\Omega \triangleq \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

skew symmetric matrix

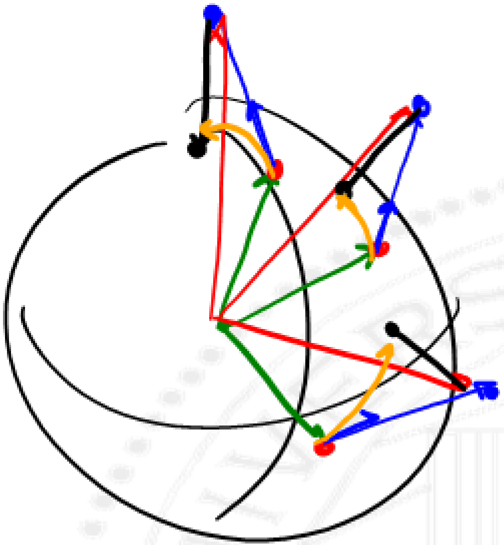
$[\omega \times]$

$$\dot{R} = -\Omega R \rightarrow R(t) = e^{\Omega t} R_0$$

$e^{\Omega t} \rightarrow I, \Omega t, \Omega t^2$

$$R_{k+1} = \frac{e^{\Omega \Delta T}}{T} R_k$$

$$\frac{\sin \|\omega\|}{\|\omega\|}$$



$$R_{k+1} = R_k + (\omega\times)\Delta T$$

$$R_{k+1} = e^{\Omega \Delta T} R_k$$

$$\Omega \triangleq \begin{bmatrix} 0 & \omega^T \\ -\omega & \omega\times \end{bmatrix}$$

$$\dot{q} = -\frac{1}{2}\Omega q$$

$e^{\Omega t}$ — closed form

Linear Dynamical Systems with Inputs and Outputs

- Inputs and Outputs: Interpretations
- Transfer Matrix
- Impulse and Step Matrices
- Examples

Inputs and Outputs (1.3)

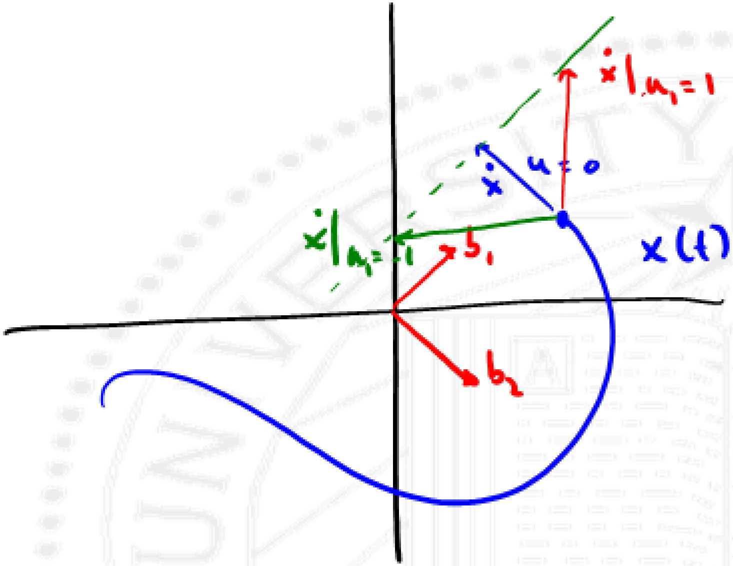
$\dot{x} = Ax + Bu$

$y = Cx + Du$

Ax — drift term, homogeneous term, unforced term, ballistic term

Bu is called: input term, driven term, forced term

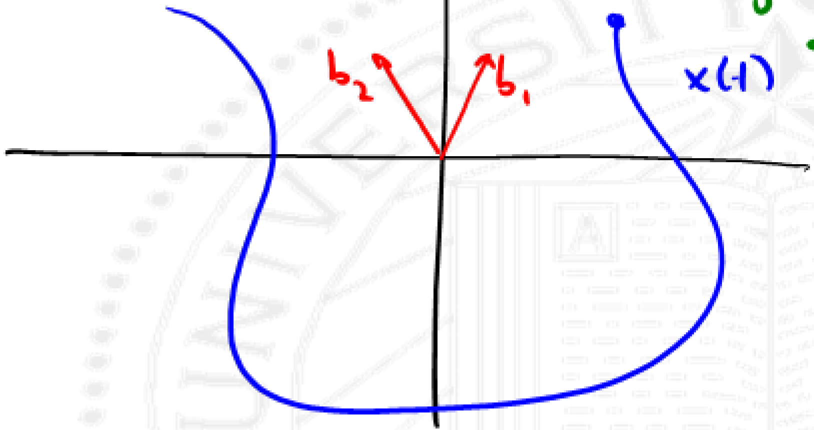
Inputs and Outputs (2.3)



Inputs and Outputs (3.3)

if span$\{b_1, b_2\} = \mathbb{R}^2$

force trajectory to any arbitrary location



Interpretations (1.2)

$\dot{x} = Ax + b_1u_1 + b_2u_2 + \ldots + b_mu_m \qquad B = [b_1 \ \ldots \ b_m]$

state derivation is the sum of the autonomous term (Ax) and one term per input (b_iu_i)

Each input u_i gives another degree of freedom for $\dot{x}$ (assuming columns of B are independent)

Interpretations (2.2)

$$\dot{x} = Ax + Bu$$

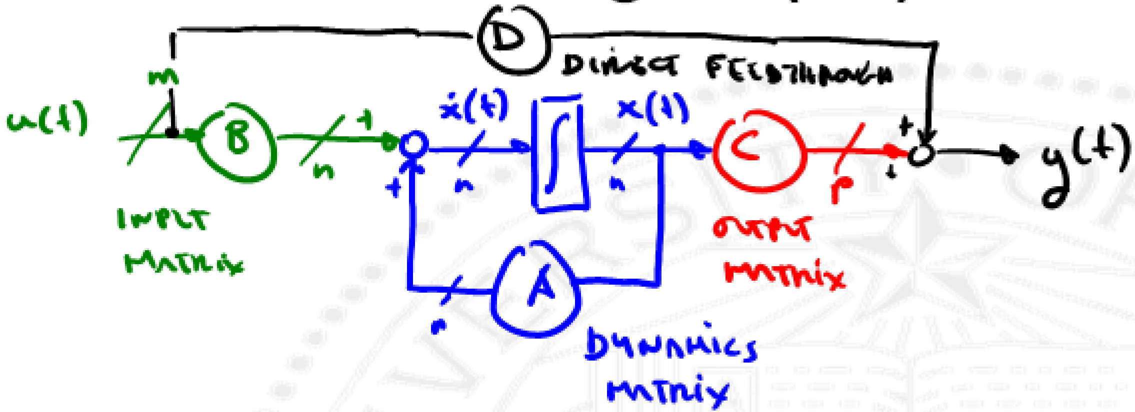
$$\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$$

↑ ith row of B.

ith state derivative is a LINEAR FUNCTION of the state x, and the input u.

Range (B) contains velocity I can hit with u.

Block Diagram (1.3)



A_{ij} gain from state x_j to integrator i

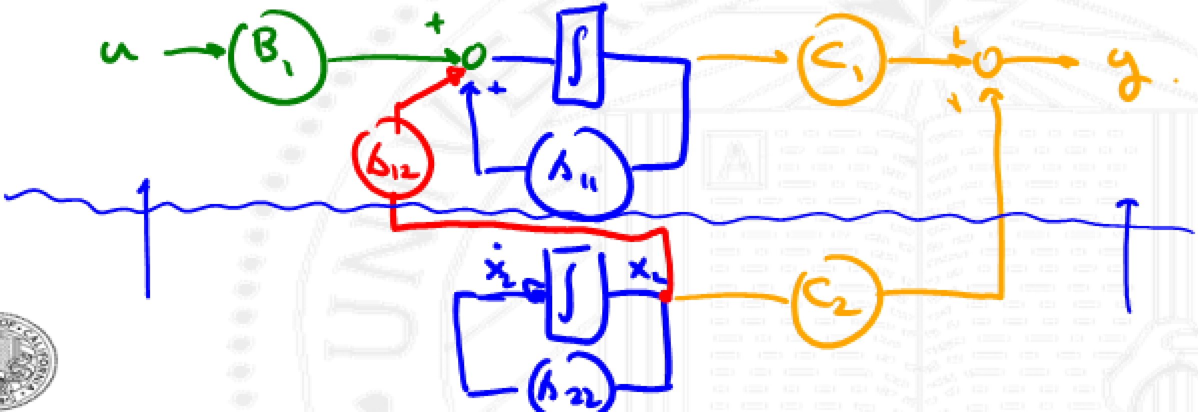
B_{ij} gain from input u_j to integrator i

C_{ij} gain from state x_j to output i

D_{ij} gain from input u_j to output i

Block Diagram (2.3)

$$\begin{bmatrix}\dot{x}_1 \\ \hline \dot{x}_2\end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22}\end{array}\right]\begin{bmatrix}x_1 \\ \hline x_2\end{bmatrix} + \begin{bmatrix}B_1 \\ \hline 0\end{bmatrix}u \qquad y = \begin{bmatrix}C_1 \\ \hline C_2\end{bmatrix}\begin{bmatrix}x_1 \\ \hline x_2\end{bmatrix}$$



Block Diagram (3.3)

Transfer Matrix (1.3)

$\mathcal{L}\{\dot{x} = Ax + Bu\} = sX(s) - x_0 = AX(s) + \underline{BU(s)}$

$X(s) = \underbrace{[sI - A]^{-1} x_0}_{\text{we know}} + \underbrace{[sI - A]^{-1} BU(s)}_{\text{product of two Laplace transforms}}$

$x(t) = e^{At}x_0 + \underline{(e^{At}B) * u(t)}$ convolution

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

Transfer Matrix (2.3)

$e^{At}x_0$ — unforced, autonomous, homogeneous, natural response.

$e^{At}B$ — input to state impulse matrix $(u_i = \delta_i)$ $\uparrow$

$[sI - A]^{-1}B$ — input to state transfer matrix

transfer function matrix

$\mathcal{L}\{y = Cx + Du\} = Y(s) = CX(s) + DU(s)$.

Transfer Matrix (3.3)

$$Y(s) = C[sI-A]^{-1}x_0 + \left[C[sI-A]^{-1}B + D\right]U(s).$$

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{i.c. response}} + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t).$$

$$H(s) \triangleq C[sI-A]^{-1}B + D$$ transfer fnct. matrix

$$h(t) = \left[Ce^{At}B + D\right]\delta(t)$$ ← impulse response matrix

dirac delta

Impulse Matrix (1.3)

$x(0) = \{0\}$ $\qquad h(t) = [Ce^{At}B + D]\,\delta(t)$

$Y(s) = H(s)\,U(s) \longrightarrow y(t) = h(t) * u(t).$

H_{ij} transfer function from input u_j to output y_i

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t-\tau)\,u_j(\tau)\,d\tau$$

h_{ij} are the impulse response from jth input to ith output

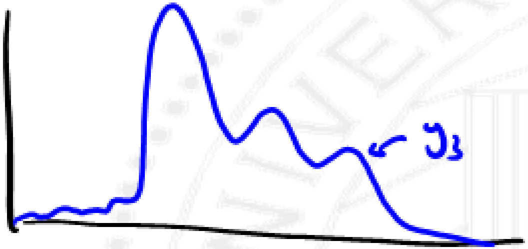
Impulse Matrix (2.3)

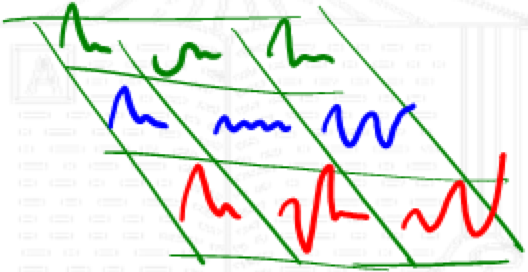
$$\begin{bmatrix}0\\ \vdots\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$h_{ij}(t)$ gives $y_i(t)$ when $u = e_j\,\delta(t)$.

$h_{35}(t)$

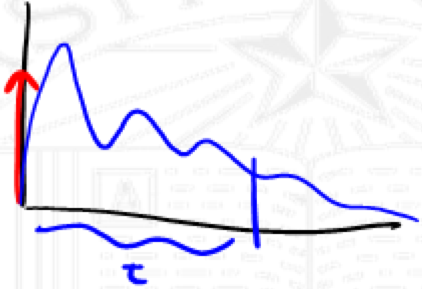
$u_5 = \uparrow$ @ $t=0$



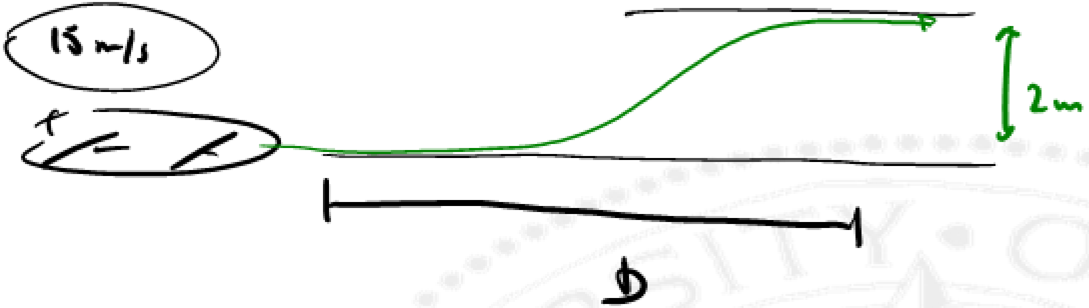


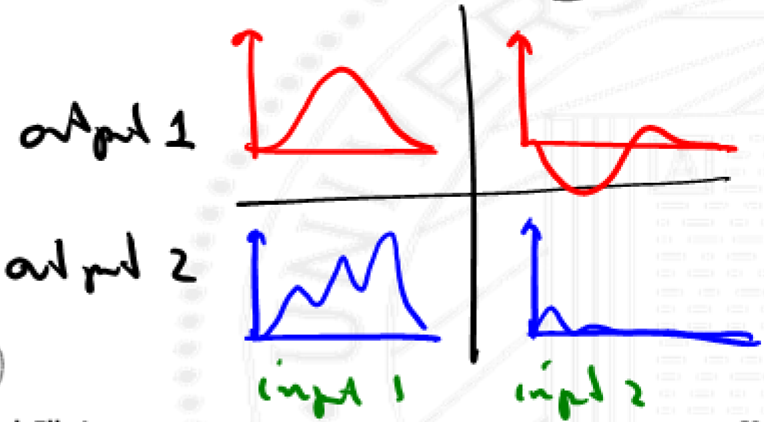
Impulse Matrix (3.3)

$h_{ij}(\tau)$ shows how dependent y_i is on input u_j τ seconds ago.



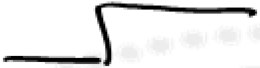
i = output
j = input
τ = time lag


15 m/s
2 m
D


output 1
output 2
input 1
input 2

Step Matrix (1.3)

$$s(t) = \int_0^t h(\tau)\,d\tau$$

input  1 $\forall\, t > 0$.

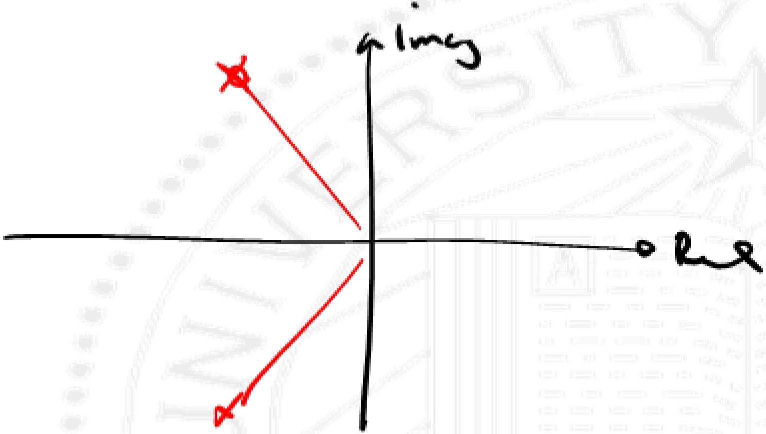
0 $t=0$

$s_{ij}(t)$ step response from j^{th} input to i^{th} output

$s_{ij}(t)$ gives $y_i(t)$ when input is e_j $\forall\, t > 0$.

Step Matrix (2.3)

$\mathrm{eig}(A) \longleftrightarrow \det(sI - A) = 0.$



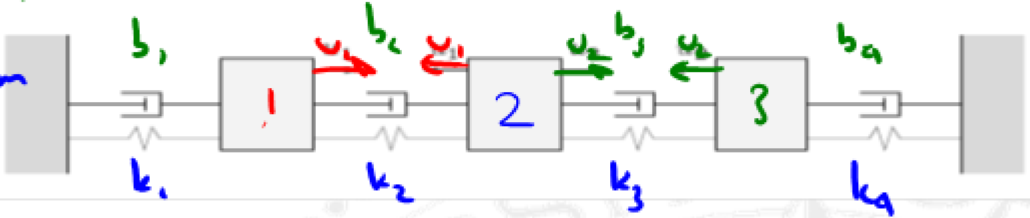
Step Matrix (3.3)

Example: Mass-Spring-Damper (1.3)

$b_i = 1$ N/(m/s)

$k_i = 1$ N/m

$m_i = 1$ kg

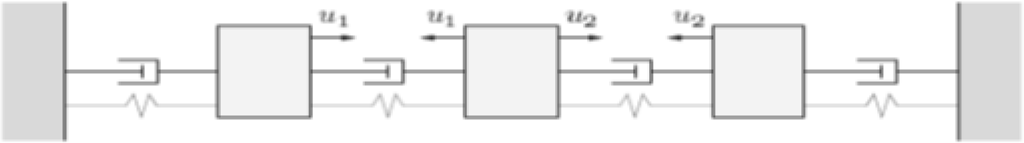


$y \in \mathbb{R}^3 \qquad \underline{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

$$\dot{\underline{x}} = \left[\begin{array}{c|c} 0 & I \\ \hline \cdots & \cdots \end{array}\right] \underline{x} + \begin{bmatrix} 0 \\ \hline [\;] \end{bmatrix} u$$

$$\begin{array}{cc} u_1 & u_2 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Example: Mass-Spring-Damper (2.3)



$$\dot{x} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

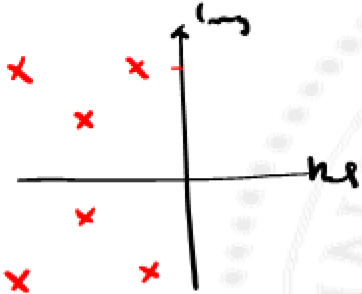
$$y = [\, I \mid 0 \,] \underline{x} + [\, 0 \,] \underline{u}$$

Example: Mass-Spring-Damper (3.3)

$\mathrm{eig}(A) = -1.71 \pm 0.36j$

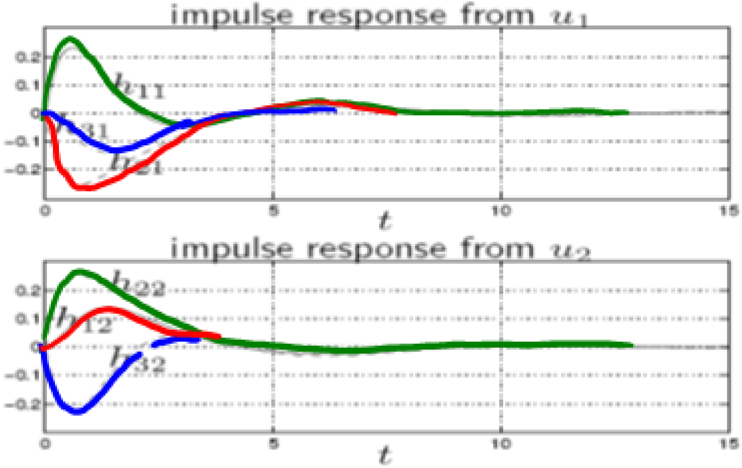
$-1 \pm j$

$-0.29 \pm 0.36j$



h_{ij}

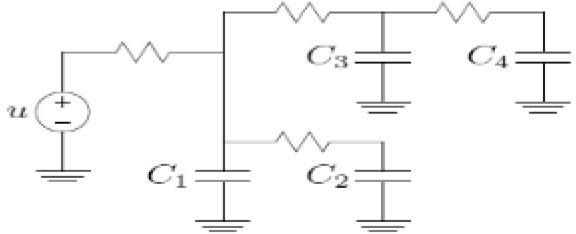
$H \in \mathbb{R}^{3 \times 2}$



u_1 [illegible], [illegible] 3 [illegible] [illegible] [illegible].

Example: Interconnect Circuit (1.3)

$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$y = Ix.$$

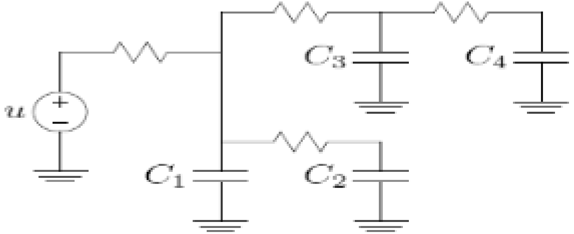
$$\mathrm{eig}(A) = \begin{bmatrix} \boxed{-0.17} \\ -0.66 \\ -2.21 \\ -3.96 \end{bmatrix}$$

$e^{\lambda t}$

Example: Interconnect Circuit (2.3)

$t=0 \rightarrow$

same voltage source



$u(t) \in \mathbb{R}$ input drive voltage

x_i voltage across capacitor C_i

output all states, $y = x$. 1V, 1F, 1Ω

Example: Interconnect Circuit (3.3)

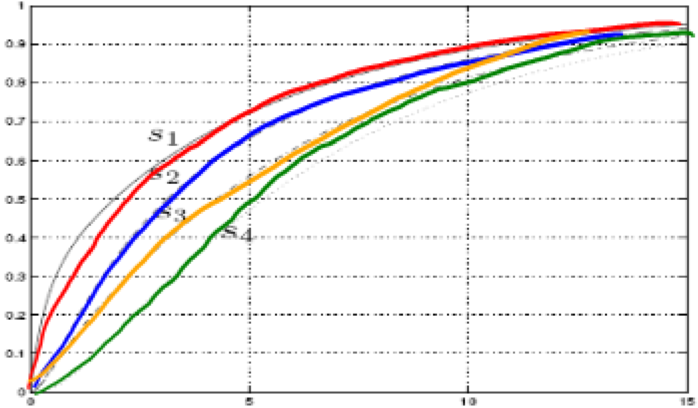
$s(t) \in \mathbb{R}^{4\times 1}$

x_1 is the fastest

shortest delay is due to x_1

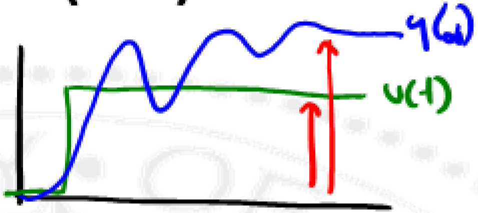
longest delay is due to x_4

dominant eigenvalue $\underline{-0.17}$.



DC or Static Gain Matrix (1.3)

Transfer function Matrix $\Big|_{s=0}$.



$$H(s)\Big|_{s=0} = C(sI-A)^{-1}B + D\Big|_{s=0} \in \mathbb{R}^{m\times p}$$

$\overset{0}{\cancel{\dot{x}}} = Ax + Bu \qquad 0 = Ax + Bu \qquad x = -A^{-1}Bu$

$y = Cx + Du \qquad y = -[CA^{-1}B + D]u$

$\frac{y}{u}\Big|_{t=\infty} \longleftrightarrow H(0)$.

DC or Static Gain Matrix (2.3)

$y(\infty) = H(0)\, u(\infty)$

step response

$$H(0) = \int_0^\infty h(t)\,dt = \lim_{t \to \infty} s(t)$$

$u(t) \to u_\infty \in \mathbb{R}^m$

$y(t) \to y_\infty \in \mathbb{R}^p$

MUST BE STABLE

$\frac{1}{(s-1)} \to H(0) = -1$

e^t

$$H(s) = \int_0^\infty e^{-st} h(t)\,dt \to s(t) = \int_0^t h(\tau)\,d\tau$$

DC or Static Gain Matrix (3.3)

$$H(0) = \begin{array}{c|c} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} & \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \\ \begin{array}{cc} u_1 & u_2 \end{array} & \end{array}$$

$$u_\infty = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_\infty = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

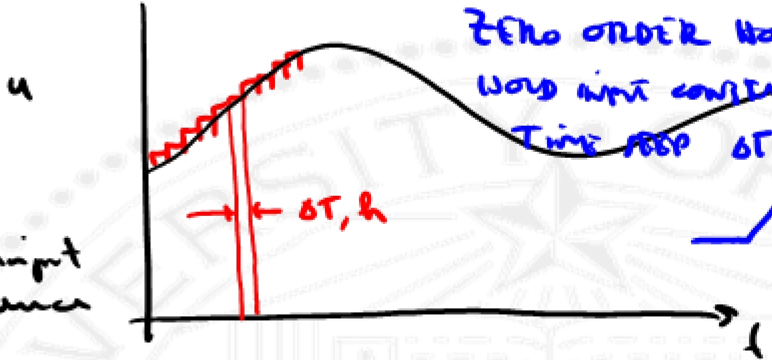
$$H(0) = C(-A)^{-1}B + \cancel{D}^{\,0} \rightarrow \begin{bmatrix} \vdots \end{bmatrix}$$

Discretization with Piecewise Constant Inputs (1.3)

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Zero order hold "ZOH"
hold input constant across time step ΔT.



$u_d : \mathbb{Z}_+ \to \mathbb{R}^m$ an input sequence

$u(t) = u_d(k)$ for $kh \le t \le (k+1)h$

$x_d(k) = x(kh)$ $\quad y_d(k) = y(kh)$ $\quad k = 0, 1, 2, \ldots$

$h > 0$ sample time (ΔT) $\quad u$ piecewise constant input

Discretization with Piecewise Constant Inputs (2.3)

$$x_d(k+1) = x((k+1)h)$$

$$= e^{Ah}x(kh) + \int_0^h e^{A\tau} B \overbrace{\left(u((k+1)h - \tau)\right)}^{\text{CONSTANT}} d\tau$$

$$= \underbrace{e^{Ah}}_{A_d} x_d(k) + \underbrace{\left[\int_0^h e^{A\tau} B\, d\tau\right]}_{B_d} u_d(k)$$

NOT AN APPROX

$$\dot{x} = Ax + Bu \longrightarrow x_d(k+1) = A_d x_d(k) + B_d u_d(k)$$

Discretization with Piecewise Constant Inputs (3.3)

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Gamma \underline{u}_k$$

$$y_k = H\underline{x}_k + D\underline{u}_k$$

$$\Phi \triangleq e^{A\Delta T}$$

$$\Gamma \triangleq \left[\int_0^{\Delta T} e^{A\tau} d\tau\right] B$$

if A is invertible:

$$\left[\int_0^{\Delta T} e^{A\tau} d\tau\right] B = \underline{A^{-1}\left(e^{A\Delta T} - I\right)B}$$

STABILITY IS PRESERVED

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Gamma \underline{u}_k$$

$$y_k = H \underline{x}_k + D \underline{u}_k$$

$$\Phi \longleftrightarrow A_d$$

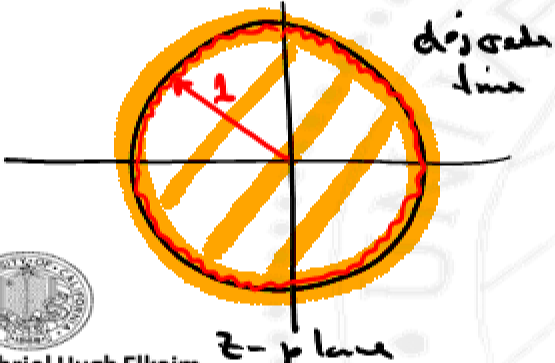
$$\Gamma \longleftrightarrow B_d$$

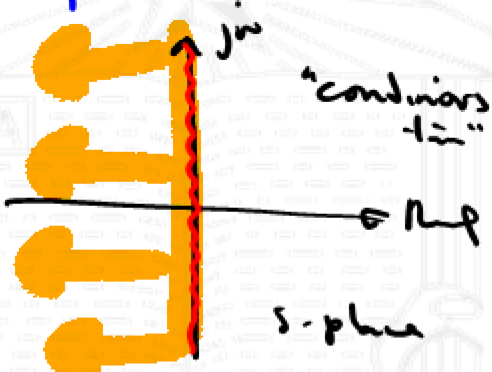
$$H \longleftrightarrow C_d$$

$\Phi \triangleq e^{A \Delta T}$ if eigenvalues of A are $(\lambda_1 \ldots \lambda_n)$

eigenvalues of Φ are $e^{\lambda_1 \Delta T}, \ldots, e^{\lambda_n \Delta T}$

if $\mathrm{Real}(\lambda_i) < 0 \longleftrightarrow |e^{\lambda_i \Delta T}| < 1$





Extensions/Variations on Discrete time systems

(1) Offsets: u_k and x_k, y_k are NOT at the same time

(2) Multirate systems: u_i and x_i, y_i are NOT at the same rate (almost always at an integer multiple of slowest one).

Dual System (1.3)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\longrightarrow \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \overbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}^{\text{NOT NECESSARILY SQUARE}} \begin{bmatrix} x \\ u \end{bmatrix}$$

$$\dot{z} = A^T z + C^T v$$
$$w = B^T z + D^T v$$

$$\longrightarrow \begin{bmatrix} \dot{z} \\ w \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix}$$

$$H(s) = C(sI - A)^{-1}B + D$$

$$H(s) = B^T(sI - A^T)^{-1}C^T + D^T \longleftarrow H(s)^T$$

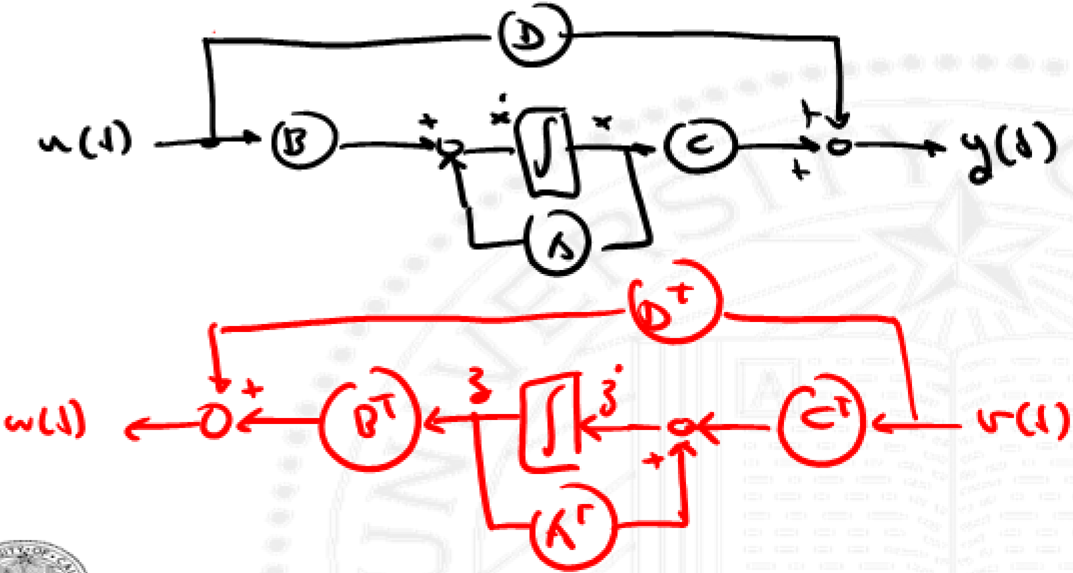
Dual System (2.3)

Single input / single output (SISO) — TF is the <u>same</u>

eigenvalues (hence all the stability properties) are also the same.

LEFT & RIGHT EIGENVECTORS <u>swap</u> places

Dual System (3.3)



Dual System via Block Diagram (1.3)

Dual System via Block Diagram (2.3)

Dual System via Block Diagram (3.3)

Causality (1.3)

$$\begin{cases} x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau \\ y(t) = C e^{At} x_0 + \int_0^t C e^{A(t-\tau)} B u(\tau) \, d\tau + D u(t) \end{cases}$$

$t \geq 0$: current state $x(t)$ and the output $y(t)$ depends <u>only</u> on past input $u(\tau)$ $\tau \leq t$.

$x(t)$ <u>cannot</u> be affected by <u>future inputs</u>.

Causality (2.3)

mapping from input to state is CAUSAL with a fixed $\underline{x_0}$

$$x(t) = f(x_0, u(0,T)).$$

fixed final state $x(T)$ for $t \le T$.

$$x(t) = e^{A(t-T)}x(T) + \int_T^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$x(t)$ depends only on $x(T), u(t,T)$

ANTI-CAUSAL

Causality (3.3)

Final state $x(T)$ and all future inputs $(u(t,T))$

tell me what the current state $x(t)$ and

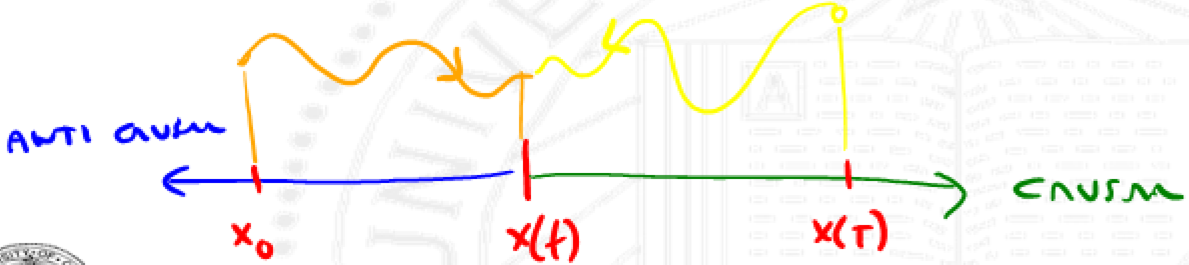
output $y(t)$ are.

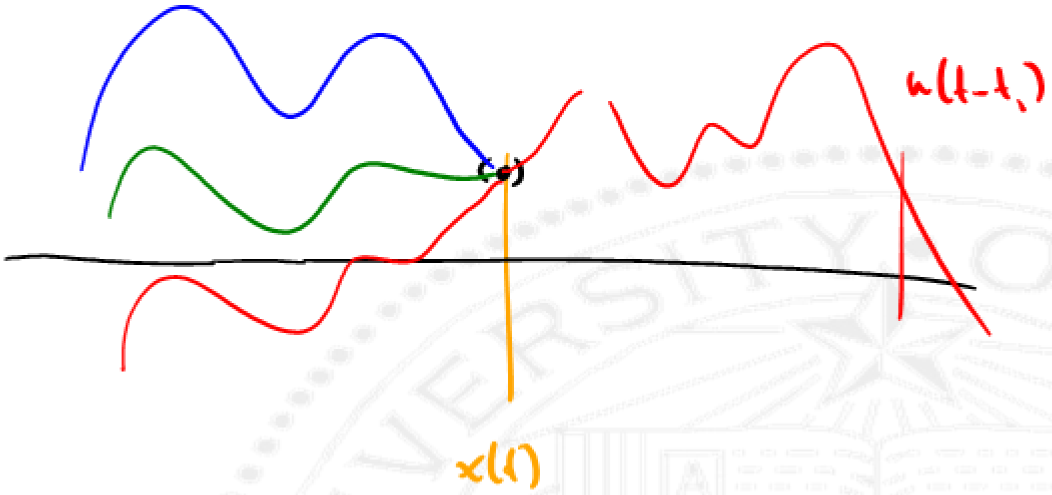
$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

CAUSALITY $\longrightarrow$ fixed x_0.

ANTI-CAUSALITY $\longrightarrow$ fixed $x(T)$





u(t-t,)
x(t)

Idea of State

$x(t)$ is called the "state" of the system @ time $= t$.

- future outputs depend <u>ONLY</u> on current state & future inputs
- future outputs depends on <u>PAST</u> inputs <u>ONLY</u> through current state
- State summarizes effects of all <u>past</u> inputs on all future outputs.
- State is a bridge between past inputs & future outputs

Change of Coordinates

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n \qquad \tilde{x} \in \mathbb{R}^n \qquad x = T\tilde{x} \quad \tilde{x} = T^{-1}x$

$y = Cx + Du$

$$\dot{(T\tilde{x})} = T\dot{\tilde{x}}$$

$$\dot{(T\tilde{x})} = AT\tilde{x} + Bu$$

$$T^{-1}\left[T\dot{\tilde{x}} = AT\tilde{x} + Bu\right]$$

$$\left[\begin{array}{l} \dot{\tilde{x}} = T^{-1}AT\tilde{x} + T^{-1}Bu \\ y = CT\tilde{x} + Du \end{array}\right]$$

Standard Forms for LDS (1.3)

$$\begin{bmatrix} \dot{x} \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ u \end{bmatrix} \longleftrightarrow \begin{bmatrix} \dot{\tilde{x}} \\ y \end{bmatrix} = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}$$

$$\underline{C(sI-A)^{-1}B + D}$$

$$\tilde{C}(sI-\tilde{A})^{-1}\tilde{B} + D$$

$$CT\left[sI-T^{-1}AT\right]^{-1}T^{-1}B + D$$

$$CTT^{-1}(sI-A)^{-1}TT^{-1}B + D$$

$$\underline{C(sI-A)^{-1}B + D}$$

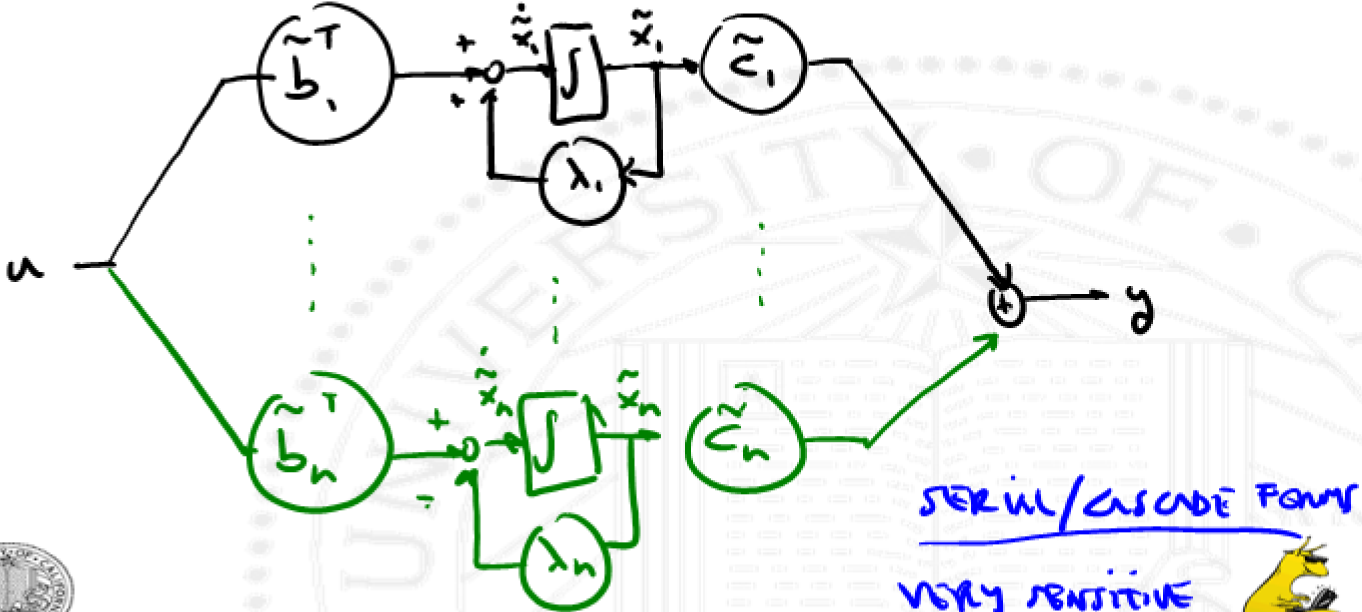
Standard Forms for LDS (2.3)

Diagonal form   find T such that $T^{-1}AT = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$

$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}$ $CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$

$\boxed{\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u}$ $\boxed{y = \sum_{i=1}^{n} \tilde{c}_i \tilde{x}_i}$

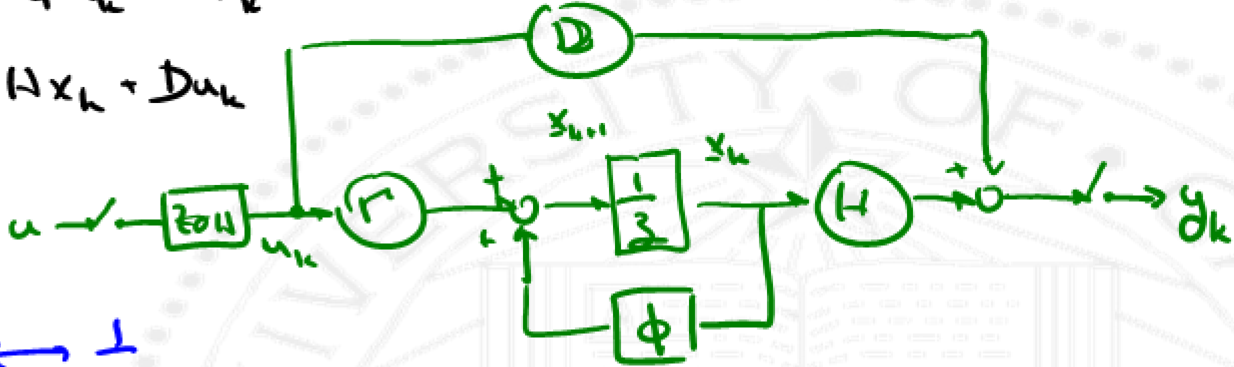
Standard Forms for LDS (3.3)



Discrete-Time Systems (1.3)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = H x_k + D u_k$$



$$\frac{1}{s} \longleftrightarrow \frac{1}{z}$$

z^{-1} – unit delay

Discrete-Time Systems (2.3)

$$x_1 = Ax_0 + Bu_0$$

$$x_2 = Ax_1 + Bu_1 = A(Ax_0 + Bu_0) + Bu_1 = A^2x_0 + ABu_0 + Bu_1$$

$$x_3 = Ax_2 + Bu_2 = A(A^2x_0 + ABu_0 + Bu_1) + Bu_2$$

$$A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2$$

$\vdots$

for $k \in \mathbb{Z}_+$ $\quad x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} B u_i$

$$y_k = CA^k x_0 + h_k * u$$ ← dynamic time convolution

Discrete-Time Systems (3.3)

$$h_k = \begin{cases} D & k=0 \\ CA^{k-1}B & k>0 \end{cases}$$

$D, CB, CAB, CA^2B, \ldots \leftarrow$ markov parameter.

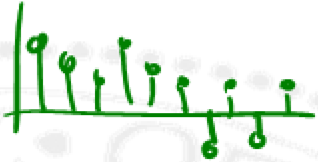
$$x_{k+10} = \phi^{10} x_k \longleftrightarrow x(t+10) = e^{A(10)} x(t)$$

$$x_{k-10} = \phi^{-10} x_k \longleftrightarrow x(t-10) = e^{-10A} x(t)$$

might not exist

FIR - Finite impulse response

$$x_k = h_1 x_{k-1} + \dots + h_n x_{k-n} + u_k$$



$$x_{k+1} = \begin{bmatrix} 0 & h_1 & \cdots & \\ & \ddots & & h_n \\ & & & 0 \end{bmatrix} \underline{x}_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u_k$$

DSP

$\downarrow$

mac.

$$\begin{bmatrix} 0.5 \\ 0.25 \\ 0.6 \\ 0.25 \end{bmatrix}$$

Z Transform (1.3)

$w \in \mathbb{R}^{p \times q}$ sequence

$w : \mathbb{Z}_+ \to \mathbb{R}^{p \times q}$

$W = \mathcal{Z}\{w\} \qquad D \subseteq \mathbb{C} \to \mathbb{C}^{p \times q}$

$$W(z) \triangleq \sum_{k=0}^{\infty} z^{-k} w(k)$$

Z Transform (2.3)

$v_k = w_{k+1} \quad k = 0, 1, 2 \ldots$

$$V(z) = \sum_{k=0}^{\infty} z^{-k} w_{k+1} = z \sum_{l=1}^{\infty} z^{-l} w_l = \underline{z W(z) - z w(0)}.$$

$$V(z) = z W(z) - z w_0 \longleftrightarrow \mathcal{L}\{\dot{x}\} = sX(s) - x_0.$$

Z Transform (3.3)

Discrete-Time Transfer Function (1.3)

$$\underline{x}_{k+1} = \Phi x_k + \Gamma u_k$$
$$y_k = H x_k + D u_k$$

$$z X(z) - z x_0 = \Phi X(z) + \Gamma U(z)$$

$$X(z) = [zI - \Phi]^{-1} z x_0 + [zI - \Phi]^{-1} \Gamma U(z).$$

$$Y(z) = \underbrace{H[zI - \Phi]^{-1} z x_0}_{\text{initial condition response.}} + \underbrace{H[zI - \Phi]^{-1} \Gamma U(z)}_{H(z).}$$

Discrete-Time Transfer Function (2.3)

$$H(z) \triangleq H(zI - \Phi)^{-1}\Gamma + D$$

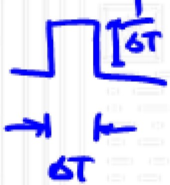
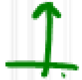
$$(zI - \Phi)^{-1} = z^{-1} I + z^{-2}\Phi + z^{-3}\Phi^2 + \ldots$$

power series expansion of the resolvent

Discrete-Time Transfer Function (3.3)

$$H(z) \triangleq H[zI - \Phi]^{-1}\Gamma + D \leftarrow \text{freq domain in } z.$$

$$h_k = \mathcal{Z}^{-1}\{H(z)\} = \underline{\text{unit}} \text{ pulse response/sequence.}$$

$u =$  $\longleftrightarrow \delta(t)$ 

$$[D \;\; H\Gamma \;\; H\Phi\Gamma \;\; H\Phi^2\Gamma \;\; \ldots] \leftarrow \text{markov parameters}$$

Questions?