Jordan Canonical Form

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$$\chi(s) = \det(sI - A)$$

λ_1 through λ_n

$A_{n-1} = -\operatorname{tr}(\lambda_i I)$ and show $\lambda_0 = \prod_{i=1}^{n} -\lambda_i$

Jordan Canonical Form

- Jordan canonical form
- Generalized modes
- Cayley-Hamilton theorem

Jordan Canonical Form (1.4)

what if $A \in \mathbb{R}^{n \times n}$ cannot be diagonalized?

any matrix $A \in \mathbb{R}^{n \times n}$ can be put into JORDAN CANONICAL FORM by a similarity transform

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_q \end{bmatrix}$$

block diagonal form

(λ_1) 1×1

$\begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}$ 2×2

$\begin{bmatrix} \lambda_3 & 1 & 0 \\ & \lambda_3 & 1 \\ 0 & & \lambda_3 \end{bmatrix}$ 3×3

Jordan Canonical Form (2.4)

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i} \qquad n = \sum_{i=1}^{q} n_i$$

Jordan Block of size n_i with eigenvalue λ_i

J_i upper bi-diagonal

J is diagonal "special case" w jordan blocks of $n_i = 1$

J is <u>unique</u> up to permutation of blocks

Jordan Canonical Form (3.4)

Can have multiple Jordan Blocks w/ same λ_i

$\lambda = (1,1,1,1)$: $J = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ 4 @ 1x1

$$\boxed{\begin{array}{c} \underline{\text{num of}} \\ N(\lambda I - J) \end{array}}$$

$J = \left[\begin{array}{cc|cc} 1 & 1 & & \\ & 1 & & \\ \hline & & 1 & 1 \\ & & & 1 \end{array}\right]$ 2 @ 2x2

$J = \left[\begin{array}{cc|cc} 1 & 1 & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right]$ 1 @ 2x2, 2 @ 1x1

$J = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$ 1 @ 4x4

$J = \left[\begin{array}{ccc|c} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ \hline & & & 1 \end{array}\right]$ 1 @ 3x3, 1 @ 1x1

$T^{-1}AT = J \therefore A = TJT^{-1} \qquad I = TT^{-1}$

Jordan Canonical Form (4.4)

$$\mathcal{X}(s) = \det(sI-A) = \det(T(sI-J)T^{-1}) = \det(T)\det(sI-J)\det(T^{-1})$$

$$= (s-\lambda_1)^{n_1}(s-\lambda_2)^{n_2}\cdots(s-\lambda_q)^{n_q}$$

distinct eigenvalues $\longrightarrow n_i = 1$ and A is fully DIAGONALIZABLE

$\dim(N(\lambda I - A))$ the number of Jordan Blocks w/ eigenvalue λ

$$\dim N(\lambda I - A)^k = \sum_{\lambda_i = \lambda} \min(k, n_i)$$

$$\left[I - \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix} \right]^k = \begin{bmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 0 \end{bmatrix}^k \leftarrow \text{shift matrix}$$

$$(\lambda_1 I - J_1)^1 = \begin{bmatrix} 0 & -1 & 0 \\ & \ddots & -1 \\ 0 & & 0 \end{bmatrix}$$

$$(\lambda_1 I - J_1)^2 = \begin{bmatrix} 0 & -1 & \\ & \ddots & -1 \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & \\ & \ddots & -1 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ & \ddots & 0 \\ & & 0 \end{bmatrix}$$

$$(\lambda_1 I - J_1)^3 = [0]$$

$$(\lambda_i I - J_j)^k = \begin{bmatrix} (\lambda_i - \lambda_j)^k & -k(\lambda_i - \lambda_j)^{k-1} & -k(k-1)(\lambda_i - \lambda_j)^{k-2} & \cdots \\ 0 & (\lambda_i - \lambda_j)^k & -k(\lambda_i - \lambda_j) & \cdots \\ 0 & 0 & (\lambda_i - \lambda_j)^k & \ddots \\ & & & \ddots \end{bmatrix}$$

$(1,1,1)$

$$\left(1I - \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}\right)^3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^3$$

rank 2

2 jordan blocks

Generalized Eigenvectors (1.3)

$$T^{-1} A T = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

$$A T_i = T_i J_i$$

$T = [T_1 \; T_2 \; \cdots \; T_q]$ where $T_i \in \mathbb{C}^{n \times n_i}$ columns of T associated w/ Jordan Block J_i

$$T_i = [v_{i1} \; v_{i2} \; \cdots \; v_{in_i}] \longrightarrow \underline{A v_{i1} = \lambda_i v_{i1}}$$

normal eigenvalue eigenvector

Generalized Eigenvectors (2.3)

The first column of each T_i block is an eigenvector associated w/ eigenvalue λ_i

$$\boxed{A v_{i_1} = \lambda_i v_{i_1}}$$

For $j = 2 \ldots n_i$

$$\boxed{A v_{i_j} = v_{i(j-1)} + \lambda_i v_{i_j}}$$

← GENERALIZED EIGENVECTORS

Generalized Eigenvectors (3.3)

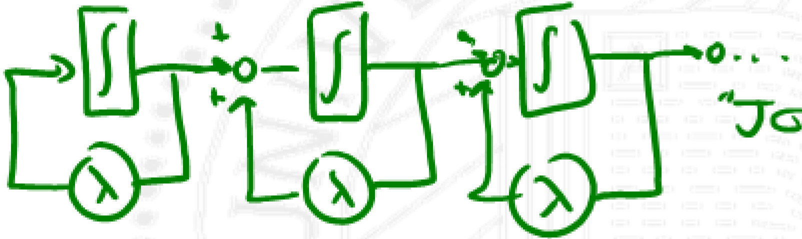
Jordan Form LDS (1.4)

$\dot{x} = Ax$ $\qquad x = T\tilde{x}$

$\dot{\tilde{x}} = J\tilde{x}$ $\qquad J \triangleq \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$

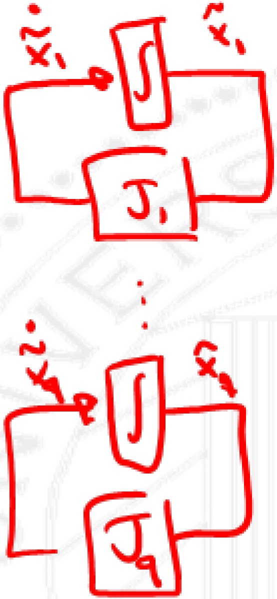
INDEPENDENT SUBSYSTEM

$\underline{\dot{\tilde{x}}_i = J_i \tilde{x}_i}$



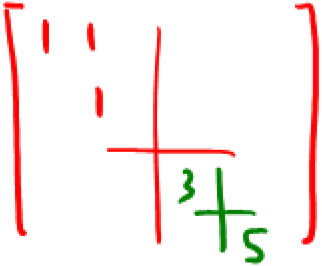
"JORDAN CHAIN"

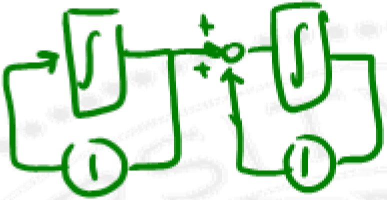
Jordan Form LDS (2.4)



$\tilde{x}_i \in R^{n_i}$

Jordan Form LDS (3.4)





Jordan Form LDS (4.4)

$$\dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} x \qquad \text{eig}(A) = (0,0,0,0)$$

<u>solution</u>: constant $\begin{vmatrix} t^0 \\ t \\ t^2 \\ t^3 \end{vmatrix}$

Resolvent and Exponential of Jordan Block (1.3)

$$(sI - J_\lambda)^{-1} = \begin{bmatrix} s-\lambda & -1 & & 0 \\ & s-\lambda & -1 & \\ & & \ddots & \ddots \\ 0 & & & s-\lambda \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (s-\lambda)^{-1} & (s-\lambda)^{-2} & \cdots & (s-\lambda)^{-k} \\ & (s-\lambda)^{-1} & \ddots & \\ & & \ddots & \end{bmatrix}$$

poln up to order k.

Resolvent and Exponential of Jordan Block (2.3)

$$(sI - J_\lambda)^{-1} = \frac{1}{s-\lambda} I + \frac{1}{(s-\lambda)^2} F_1 + \dots + \frac{1}{(s-\lambda)^k} F_{k-1}$$

$$F_i = \begin{bmatrix} 0 & & 1 & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

1s along i^{th} super diagonal

$$e^{J_\lambda t} = e^{\lambda t}\left[I + tF_1 + \frac{t^2}{2!} F_2 + \dots + \frac{t^{k-1}}{(k-1)!} F_{k-1} \right]$$

Resolvent and Exponential of Jordan Block (3.3)

$$e^{J_\lambda t} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & t \\ & & & & 1 \end{bmatrix}$$

e^{Jt} JORDAN BLOCKS

REPEATED POLES IN RESOLVENT

TAKES THE FORM OF $t^p e^{\lambda t}$

in e^{At}.

Generalized Modes (1.3)

$\dot{x} = Ax \qquad x(0) = a_1 v_{i1} + a_2 v_{i2} + \dots + a_{n_i} v_{i n_i} = \underbrace{T_i a}_{\tilde{x}_i(0)}$

$x(t) = T e^{J t} \tilde{x}(0) = \sum_{i=1}^{q} T_i e^{J_i t} a$

- Trajectories of $x(t)$ stay in the span of $\{v_{i_1} \dots v_{i n_i}\}$
- Coefficients have the form of $p(t) e^{\lambda t}$

  $p(t)$ polynomial in t.

- Solutions of $p(t) e^{\lambda t}$ – "generalized modes"

Generalized Modes (2.3)

for a generic $x(0)$

$$x(t) = e^{At}x(0) = Te^{Jt}T^{-1}x_0 = \sum_{i=1}^{q} T_i e^{J_i t}(S_i^T x(0))$$

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix} \leftarrow \text{LEFT EIGENVECTORS}$$

$$T = \begin{bmatrix} T_1 & \cdots & T_q \end{bmatrix}$$

Generalized Modes (3.3)

All solutions of $\dot{x} = Ax$ are LINEAR COMBINATIONS of the generalized modes

(1) Not ALL matrices are diagonalizable

(2) Distinct eigenvalues — it is diagonalizable $T^{-1}AT = \Lambda$

(3) All matrices can be put into JORDAN FORM

(4) Multiple poles in the resolvent $t^p e^{\lambda t}$ response.

Cayley-Hamilton Theorem (1.4)

$p(s) = a_0 + a_1 s + a_2 s^2 + \ldots + a_k s^k$ polynomial

$p(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_k A^k$ matrix analog

<u>Cayley Hamilton</u>

for any $A \in \mathbb{R}^{n \times n}$ $\quad \mathcal{X}(A) = 0 \quad \mathcal{X}(s) = \det(sI - A)$

WRONG PROOF

$\mathcal{X}(A) = \det(AI - A) = \cancel{\det(A - A)} = \det(0) = 0$

Cayley-Hamilton Theorem (2.4)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \chi(s) = \lambda^2 - 5\lambda - 2 = \det(\lambda I - A) = 0.$

$\underline{\chi(A) = A^2 - 5A - 2I} = \phi.$

$$\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Cayley-Hamilton Theorem (3.4)

for every $p \in \mathbb{Z}_+$ we have

$$A^p \in \text{span}\{I, A, A^2 \ldots A^{p-1}\}$$

any power of A can be expressed as a linear combination of $I, A, A^2 \ldots A^{p-1}$.

Cayley-Hamilton Theorem (4.4)

divide $\chi(s)$ into $s^p \longrightarrow s^p = q(s)\,\chi(s) + \underbrace{r(s)}_{\text{degree} < n}$

$r(s) = \alpha_0 + \alpha_1 \lambda + \dots + \alpha_{n-1} \lambda^{n-1}$

$$A^p = q(A)\,\cancel{\chi(A)}^{\,0} + r(A) = r(A)$$

$$= \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

coef. of $s^p/\chi(s)$

Proof of Cayley-Hamilton (1.3)

$$\chi(A) = A^n + a_{n-1}A^{n-1} + \dots + a_0 I = 0 \qquad A^p \to \underline{p = -1}$$

$$a_0 I = -A^n - a_{n-1}A^{n-1} - \dots - a_1 A$$

$$I = \frac{-1}{a_0}A^n - \frac{a_{n-1}}{a_0}A^{n-1} - \dots - \frac{a_1}{a_0}A$$

$$I = \underbrace{\left[\frac{-1}{a_0}A^{n-1} - \frac{a_{n-1}}{a_0}A^{n-2} \dots - \frac{a_1}{a_0}I\right]}_{A^{-1}} A$$

Proof of Cayley-Hamilton (2.3)

$$A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \ldots - \frac{1}{a_0} A^{n-1}$$

if $a_0 = 0$ ~~and~~ invertible

Inverse of A is a linear combination of A^k $k = 0 \ldots n-1$

FAST METHOD of solving $AX = B$

Proof of Cayley-Hamilton (3.3)

Assume $A \in \mathbb{R}^{n\times n}$ is diagonalizable $\longrightarrow T^{-1}AT = \Lambda$

$\chi(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

$$\chi(A) = \chi(T^{-1}AT) = T\,\underset{\overset{\parallel}{0}}{\chi(\Lambda)}\,T^{-1}$$

$$\chi(\Lambda) = (\Lambda-\lambda_1 I)(\Lambda-\lambda_2 I)\cdots(\Lambda-\lambda_n I)$$

$$\left(\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} - \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{bmatrix}\right)\left(\begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix} - \begin{bmatrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{bmatrix}\right)\cdots\Big($$

$$\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} - \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{bmatrix}\right) = \begin{bmatrix} 0 & & & & \\ & \lambda_2-\lambda_1 & & & \\ & & \lambda_3-\lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_n-\lambda_1 \end{bmatrix}$$

$$\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} - \begin{bmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda_1-\lambda_2 & & & & \\ & 0 & & & \\ & & \lambda_3-\lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_n-\lambda_2 \end{bmatrix}$$

$$\left(\begin{bmatrix}0&&&\\&*&&\\&&\ddots&\\&&&*\end{bmatrix}\begin{bmatrix}*&&&\\&0&&\\&&*&\\&&&\ddots\\&&&&*\end{bmatrix}\cdots\begin{bmatrix}*&&&\\&\ddots&&\\&&*&\\&&&0\end{bmatrix}\right)=\begin{bmatrix}0&&\\&\ddots&\\&&0\end{bmatrix}$$

$T^{-1} A T = J$ $\qquad \chi(s) = (s-\lambda_1)^{n_1} \cdots (s-\lambda_q)^{n_q}$

$\chi(J_i) = 0$

$$\chi(J_i) = (J_i - \lambda_1 I)^{n_1}(J_i - \lambda_2 I)^{n_2}(\dots)\underbrace{\begin{bmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{bmatrix}^{n_i}}_{(J_i - \lambda_i I)^{n_i}}(\dots)(J_i - \lambda_q I)^{n_q}$$

$$\begin{bmatrix}0 & 1 & \\ & \ddots & \ddots \\ & & 0\end{bmatrix}^{k} = \begin{bmatrix}0 & & \\ & \ddots & \\ & & 0\end{bmatrix}$$

Questions?