Eigenvectors and Diagonalization

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Eigenvectors and Diagonalization

- Eigenvectors
- Dynamic interpretation—invariant sets
- Complex eigenvectors and invariant planes
- Left eigenvectors
- Diagonalization
- Modal Form
- Discrete-time stability

Eigenvectors and Eigenvalues (1.3)

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0 .$$

eigenvalues are points where the RESOLVANT is undefined

$\exists\, v \in \mathbb{C}^n$ such that $(\lambda I - A)v = 0$.

$$\boxed{\lambda v = Av}$$

any such v is called a RIGHT EIGENVECTOR of A corresponding to eigenvalue λ.

Eigenvectors and Eigenvalues (2.3)

$w \in \mathbb{C}^n$ such that $w^T(\lambda I - A) = 0$

$$\boxed{w^T A = \lambda w^T}$$

such a w is a <u>LEFT EIGENVECTOR</u> of A corresponding to eigenvalue λ.

$A \in \mathbb{R}^{n \times n}$ $\quad$ $\lambda, v, w \in \mathbb{C}^{(n)}$

$v \in \mathbb{C}^n$ $\quad$ $\lambda \in \mathbb{C}$ $\quad$ $[Av = \lambda v]$ $\quad$ $\bar{v}, v^*, v^H$

$\bar{v} \in \mathbb{C}^n$ $\quad$ $\bar{\lambda} \in \mathbb{C}$ $\quad$ $\bar{A}\bar{v} = \bar{\lambda}\bar{v}$

$A\bar{v} = \bar{\lambda}\bar{v}$

MATLAB $'$ $\rightarrow$

Eigenvectors and Eigenvalues (3.3)

$Av = \lambda v$ $\dot{x} = Ax$ $x_0 = v$

$\dot{x} = Ax = Av = \lambda v$

$\dot{x} = \lambda v$

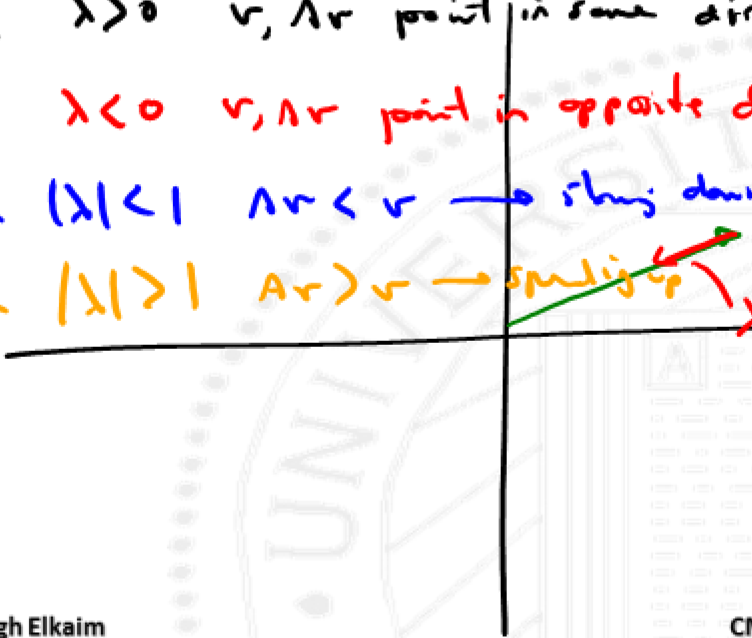
Scaling Interpretation (1.2)

$\lambda \in \mathbb{R}$ $\lambda > 0$ $v, \lambda v$ point in same direction

$\lambda \in \mathbb{R}$ $\lambda < 0$ $v, \lambda v$ point in opposite directions

$\lambda \in \mathbb{R}$ $|\lambda| < 1$ $\lambda v < v$ → shrinking down

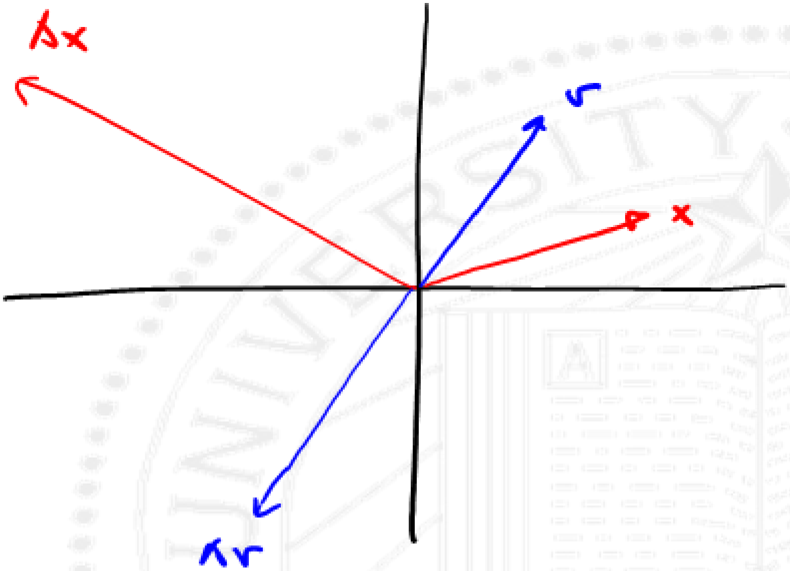
$\lambda \in \mathbb{R}$ $|\lambda| > 1$ $\lambda v > v$ → swelling up



λv growing with t.

λv going towards 0.

Scaling Interpretation (2.2)



Dynamic Interpretation (1.2)

$x(t) = e^{\lambda t} v$ "mode" of the system

$Av = \lambda v \quad v \neq 0.$

$\dot{x} = Ax \quad x(0) = v \qquad \boxed{x(t) = e^{\lambda t} v}$

$x(t) = e^{At} v = \left[I + At + \frac{A^2 t^2}{2!} + \dots\right] v$

$= v + Avt + A(Avt)\frac{t}{2} + \dots$

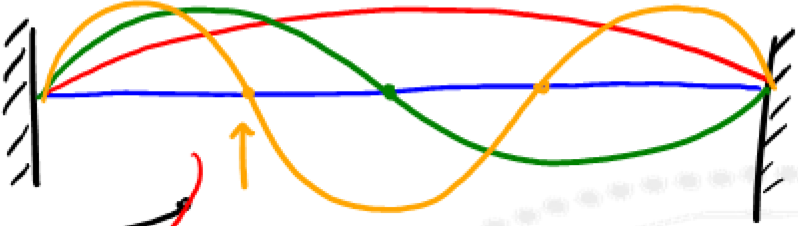
$= v + \lambda t v + \frac{\lambda^2 t^2}{2!} v + \dots = e^{\lambda t} v.$

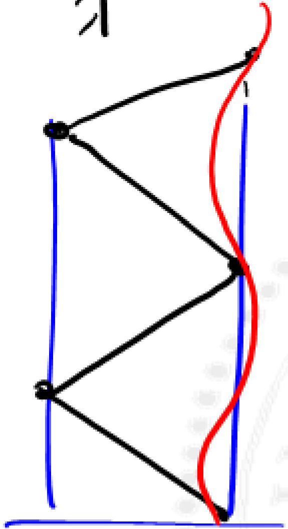
Dynamic Interpretation (2.2)

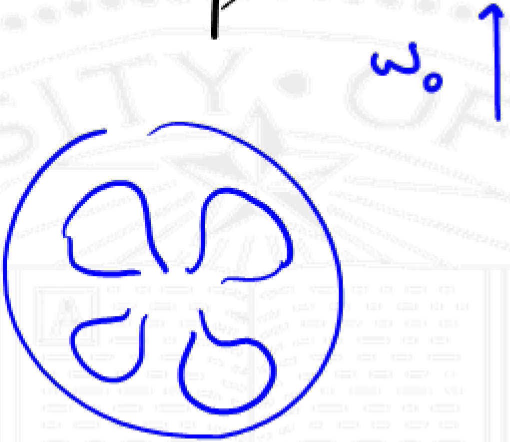
$x(t) = e^{\lambda t} v$ ← system mode

if the initial state $x(0) = v$ resulting motion is always on a line spanned by v

$x(t) = e^{\lambda t} v$ is mode associated w/ λ_i





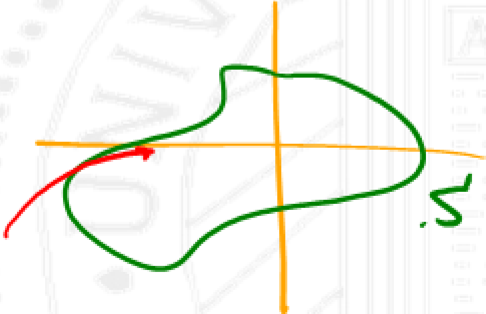

ω_0

Invariant Sets (1.3)

a set $S \subseteq \mathbb{R}^n$ <u>invariant</u> under $\dot{x} = Ax$

is whenever $x(t) \in S$ then $x(\tau) \in S$ $\forall$ $\tau \geq t$.

Once a trajectory enters S it remains in S.



Invariant Sets (2.3)

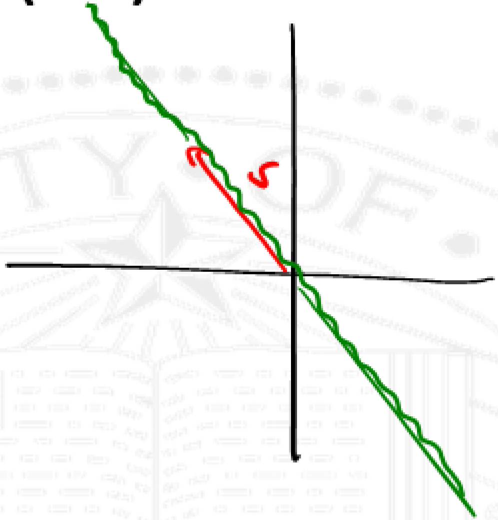
Vector Field Interpretation

$Av = v\lambda \quad v \neq 0 \quad \lambda \in \mathbb{R}$

line $\{tv \mid t \in \mathbb{R}\}$ is invariant

$\lambda < 0$ line segment

$\{tv \mid 0 \leq t < a\}$ invariant

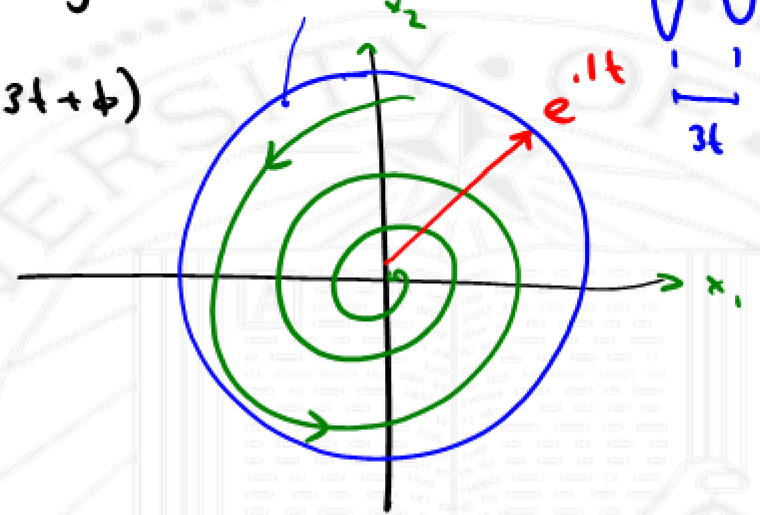


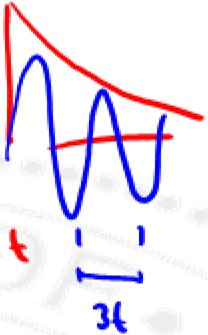
Invariant Sets (3.3)

$\dot{x} = Ax$

$\lambda = -0.1 \pm 0.3j$

$e^{0.1t} \cos(3t + \phi)$





Complex Eigenvectors (1.2)

$Av = \lambda v \quad v \neq 0 \quad \lambda$ complex

for $a \in \mathbb{C}$ (complex trajectory) $\underbrace{ae^{\lambda t} v}_{\text{complex, oscillatory behavior}}$ satisfies $\dot{x} = Ax$

$x(t) = \mathrm{Re}\left(a e^{\lambda t} v\right)$ $\quad e^{\lambda t} = e^{\sigma t}\left[\cos \omega t + j \sin \omega t\right]$

$$x(t) = e^{\sigma t} \underbrace{\begin{bmatrix} v_{re} & v_{im} \end{bmatrix}}_{\text{describes a plane in } \mathbb{R}^n} \underbrace{\begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}}_{\text{rotation matrix}} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\text{constants}}$$

Complex Eigenvectors (2.2)

$v = v_{real} + j v_{imag}$

$\lambda = \sigma + j\omega$

$a = \alpha - j\beta$

trajectory stays in an invariant plane $\mathrm{span}\{v_{real}, v_{imag}\}$

σ gives logarithmic growth/decay

ω gives the angular velocity of rotation in the plane.

Dynamic Intepretation: Left Eigenvectors (1.2)

$w^T A = \lambda w^T \quad w \neq 0 \quad w^T x(t) \leftarrow$ scalar function of time.

$$\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x \qquad \dot{x} = Ax$$

$$= \lambda w^T x = \lambda (w^T x).$$

$w^T x$ satisfies a D.E. $\frac{d}{dt}(w^T x) = \lambda(w^T x)$

$$\boxed{w^T x(t) = e^{\lambda t} w^T x(0)}$$

$\dot{x} = Ax$

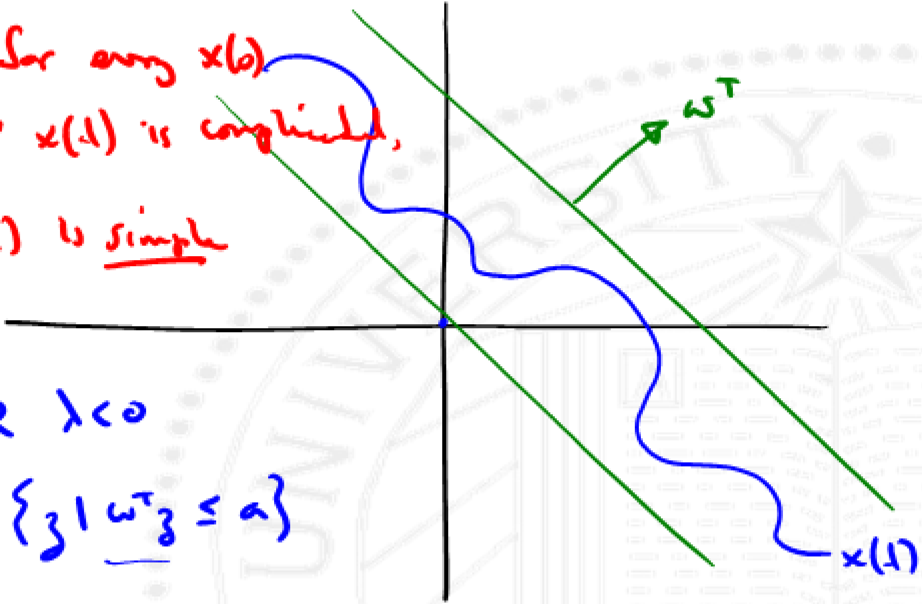
$x(0) = x_0$.

Dynamic Intepretation: Left Eigenvectors (2.2)

holds for any $x(0)$

even if $x(1)$ is complicated,

$w^T x(1)$ is <u>simple</u>



$\lambda \in \mathbb{R}$ $\lambda < 0$

$\{z \mid \underline{w^T z} \le a\}$

Eigenvector Summary

- Right eigenvectors are initial conditions such that the resulting motion of $\dot{x} = Ax$ is very simple.

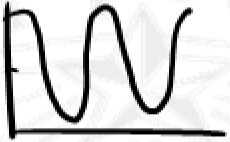
- Left eigenvectors give linear functions of the state that are simple for all $x(0)$.

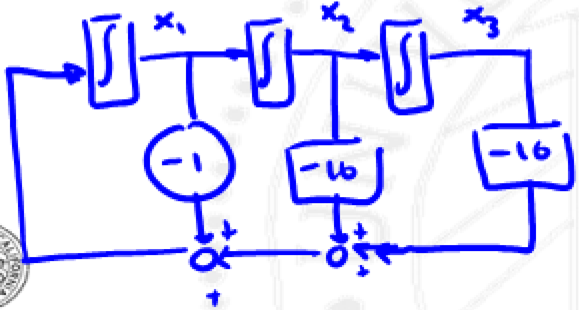
Eigenvalue/vector example (1.4)

$$\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}$$

e^{-t}

$\cos\sqrt{10}\,t$





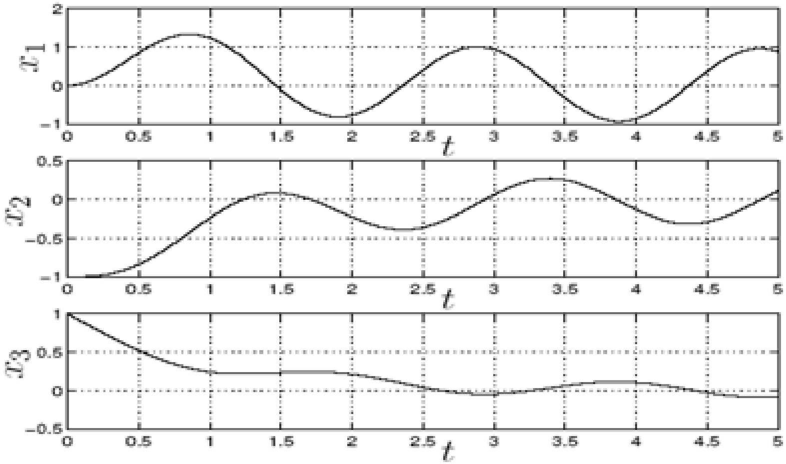
$\lambda : -1, \pm j\sqrt{10}$

Eigenvalue/vector example (2.4)

$x(0) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ e^{-t}

$\cos(\sqrt{10}\, t)$

$\lambda = -1$ $w = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}$

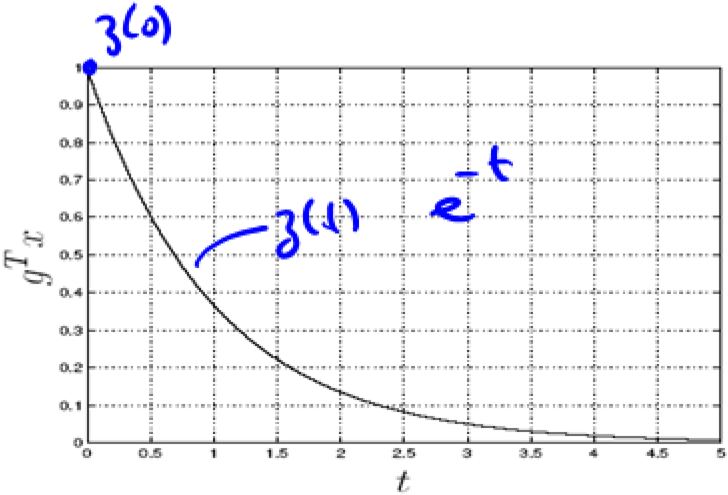


Eigenvalue/vector example (3.4)

$z(t) = [0.1 \quad 0 \quad 1]\, x(t)$

$z(0) = 0.1\, x_{1,0} + 0\, x_{2,0} + 1\, x_{3,0}$

$z(t)$

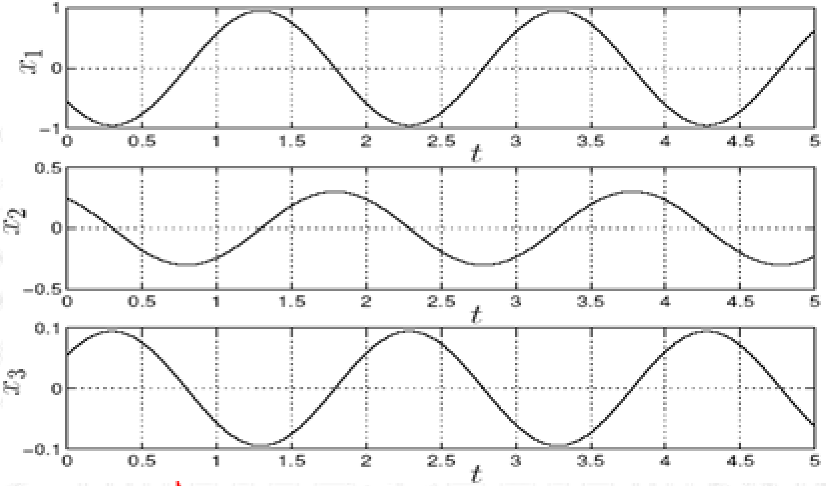


Eigenvalue/vector example (4.4)

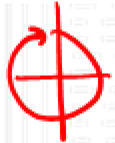
$\lambda = j\sqrt{10}$

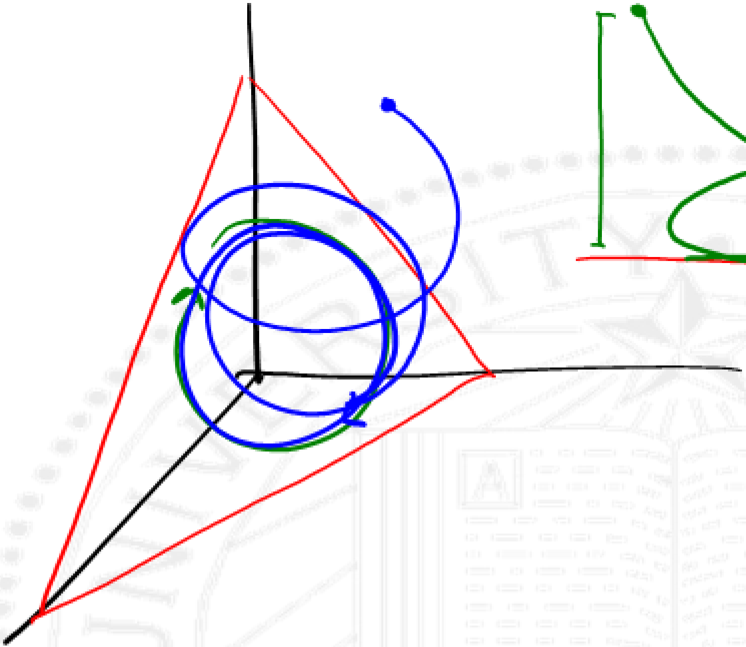
$$v = \begin{bmatrix} 0.559 + j0.771 \\ 0.244 + j0.175 \\ 0.055 - j0.077 \end{bmatrix}$$

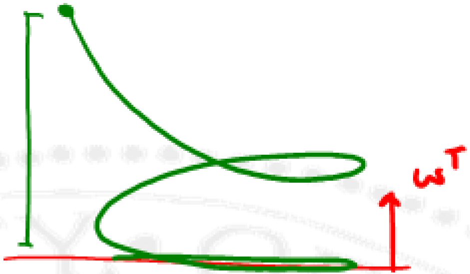
$$v_{real} = \begin{bmatrix} -0.559 \\ 0.244 \\ 0.055 \end{bmatrix} \quad v_{imag} = \begin{bmatrix} 0.771 \\ 0.175 \\ 0.077 \end{bmatrix}$$



$x(0) = v_{real}$







w^T

Example: Markov Chain (1.3)

$$p(t+1) = P\,p(t)$$

$$p_i(t) = \text{prob}(z(t)=i) \quad \text{so: } \sum_{i=1}^{n} p_i(t) = 1.$$

$$P_{ij} \triangleq \text{prob}(z(t+1)=i \mid z(t)=j) \quad \text{so: } \sum_{i=1}^{n} P_{ij} = 1.$$

$$\underbrace{[1 \; 1 \; \cdots \; 1]}_{w^T = \mathbb{1}^T} \overset{n\times n}{P} = [1 \; 1 \; \cdots \; 1] \qquad \det(\overset{1}{\cancel{\lambda}} I - P) = 0.$$

$$\lambda = 1 \qquad Pv = v \quad v \neq 0.$$

Example: Markov Chain (2.3)

choose to normalize v : $\sum_{i=1}^{n} v_i = 1$

what is v? $\lambda = 1$.

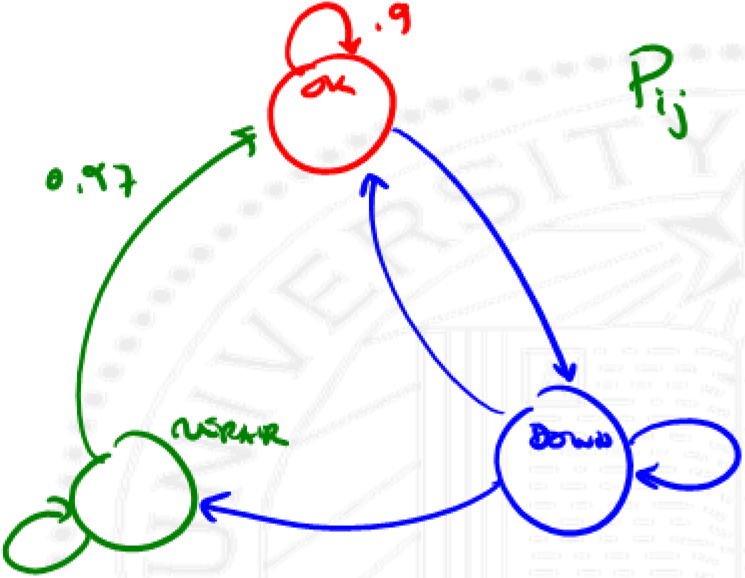
$p(0) = v$ $\quad Pv = v$

$p(t+1) = v$

<u>if</u> v is unique → STEADY STATE DISTRIBUTION of the MARKOV CHAIN

v – equilibrium distribution

Example: Markov Chain (3.3)



P_{ij}

$s = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$

Diagonalization (1.3)

$v_1 \dots v_n$ linearly independent (right) eigenvectors of $A \in \mathbb{R}^{n \times n}$

$$A v_i = \lambda_i v_i \qquad i = 1..n$$

$$A \underbrace{\begin{bmatrix} v_1 \dots v_n \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} v_1 \dots v_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$T^{-1} A T = T^{-1} T \Lambda \longrightarrow \boxed{T^{-1} A T = \Lambda}$$

Example: Markov Chain (1.4)

$$T^{-1}T = \sum_{i=1}^{j} w_i^T v_j = I \qquad T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

$$T T^{-1} = \sum_{i=1}^{j} \underbrace{v_i w_j^T}_{\text{diad, rank 1 matrix}} = I$$

Diagonalization (2.3)

Call $A \in \mathbb{R}^{n \times n}$ "diagonalizable"

- There exists a T such that $T^{-1}AT = \Lambda$
  (Λ diagonal)

- A has a set of linearly independent eigenvectors
  $T \triangleq [v_1 \dots v_n]$

$A \not\to \Lambda$ "A" defective

Diagonalization (3.3)

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\chi(s) = s^2$ $\lambda = (0, 0)$

$Av = 0v$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ <u>all</u> eigenvectors when $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ $v_1 \neq 0$.

A is defective or not diagonalizable cannot make

v_1 v_2 independent

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \chi(s) = \frac{1}{(s-1)^3} \qquad \lambda = (1,1,1)$$

$$Av = \lambda v \rightarrow Iv = \lambda v \rightarrow \boxed{v = v}$$

all v are valid eigenvectors.

Not all matrices are diagonalizable

Distinct Eigenvalues

FACT: if A has distinct eigenvalues

$\lambda_i \neq \lambda_j \quad \forall\, i \neq j$

then A is always diagonalizable

Diagonalization and Left Eigenvectors (1.2)

$T^{-1} A T T^{-1} = \Lambda T^{-1}$

$T^{-1} A = \Lambda T^{-1}$

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

$w_i^T v_j \triangleq \delta_{ij}$ $\begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$w_1^T \ldots w_n^T$ are rows of T^{-1}

$w_i^T A = \lambda_i w_i^T$

rows of T^{-1} are the linearly independent LEFT EIGENVECTORS of A.

Diagonalization and Left Eigenvectors (2.2)

$\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{T}$ ← right eigenvectors

$\underline{w_i^T v_j = \delta_{ij}}$ $\begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

BI-ORTHOGONALITY

$\underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}}_{T^{-1}}$ ← LEFT eigenvectors

$A = [a_1 \cdots a_n] \in \mathbb{R}^{n\times n}$ square, non-singular

$$A^{-1} = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} \longrightarrow b_i^T a_j = \delta_{ij}$$

$$x = AA^{-1}x = [a_1 \cdots a_n]\begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} x = (b_1^T x)a_1 + \cdots + (b_n^T x)a_n$$

$$x = A^{-1}Ax = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix}[a_1 \cdots a_n]x = \begin{bmatrix} (a_1 x)\, b_1^T \\ \vdots \\ (a_n x)\, b_n^T \end{bmatrix}$$

Modal Form (1.3)

Suppose A is diagonalizable by T

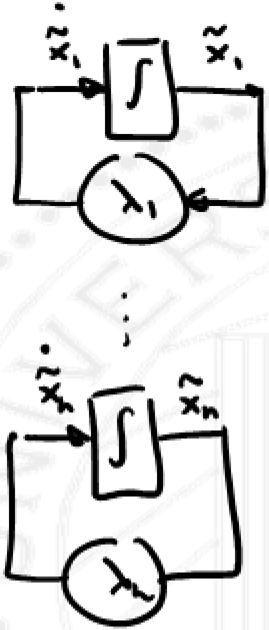
define a new set of coordinates $x = T\tilde{x}$

$$\dot{x} = Ax \rightarrow \dot{(T\tilde{x})} = AT\tilde{x} \rightarrow T^{-1}T\dot{\tilde{x}} = T^{-1}AT\tilde{x}$$

$$\boxed{\dot{\tilde{x}} = T^{-1}AT\tilde{x}}$$

$$\boxed{\dot{\tilde{x}} = \Lambda\tilde{x}} \leftarrow \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Modal Form (2.3)



n independent modes

$$\boxed{\tilde{x}_i(t) = e^{\lambda_i t}\,\tilde{x}_i(0).}$$

MODAL FORM

Modal Form (3.3)

$\dot{x} = Ax \qquad T^{-1}AT = \Lambda$

$x(t) = e^{At}x(0)$

$e^{At} = Te^{\Lambda t}T^{-1}$

$= Te^{\Lambda t}T^{-1}x(0)$

$= \sum_{i=1}^{n} e^{\lambda_i t}(w_i^T x(0))v_i$

reconstitute <u>state</u> from its modal responses.

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} (w_i^T x(0)) v_i$$

v_i ← "modes" or "mode shapes" of the system

w_i^T ← left eigenvectors decompose $x(0)$ into modal coordinates

$e^{\lambda_i t}$ propagates the ith mode.

Real Modal Form (1.3)

When eigenvalues of A are complex, then T is also complex.

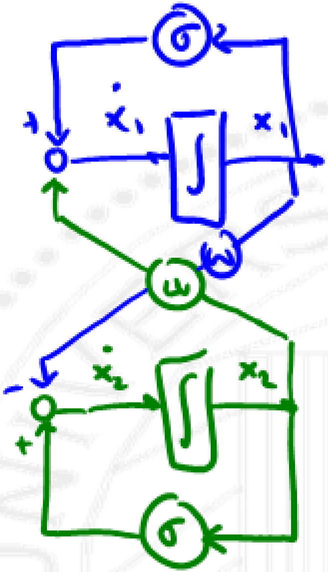
$$S^{-1}AS = \text{diag}\left(\Lambda_r, \begin{bmatrix} \sigma_{r+1} & \omega_{r+1} \\ -\omega_{r+1} & \sigma_{r+1} \end{bmatrix}, \ldots, \begin{bmatrix} \sigma_n & \omega_n \\ -\omega_n & \sigma_n \end{bmatrix}\right)$$

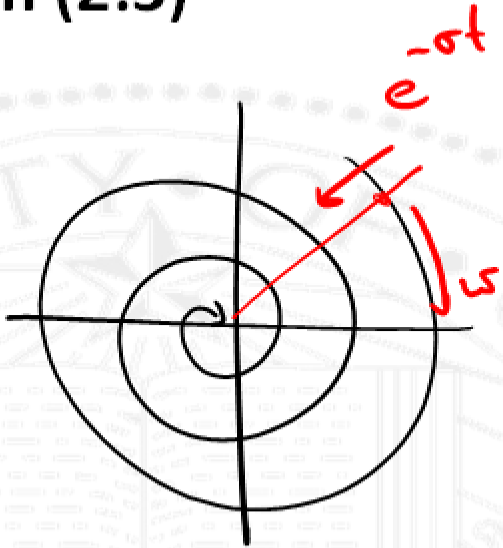
$$\det\left(sI - \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}\right) = 0 \longrightarrow (\lambda - \sigma)^2 + \omega^2 \longrightarrow \lambda_i = \sigma \pm j\omega$$

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_r & & & \\ & & & \begin{bmatrix} \sigma_1 & \omega \\ -\omega & \sigma \end{bmatrix} & & \\ & 0 & & & \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} & \\ & & & & & \ddots \end{bmatrix}$$

Real Modal Form (2.3)

$\dot{x} = Ax$





Real Modal Form (3.3)

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$T^{-1}AT = \Omega \quad \therefore \quad A = T\Omega T^{-1}$

resolvent: $(sI-A)^{-1} = (\underset{TT^{-1}}{sI} - T\Omega T^{-1})^{-1} = \left[sTT^{-1} - T\Omega T^{-1}\right]^{-1}$

$= \left[\underset{A}{T}\,\underbrace{(sI-\Omega)}_{B}\,\underset{C}{T^{-1}}\right]^{-1}$

$= T\left[sI-\Omega\right]^{-1}T^{-1}$

$= \left[sI-A\right]^{-1}$

$$[sI - A]^{-1} = T(sI - \Lambda)^{-1} T^{-1}$$

$$= T \begin{bmatrix} (s-\lambda_1) & & 0 \\ & \ddots & \\ 0 & & (s-\lambda_n) \end{bmatrix}^{-1} T^{-1}$$

$$= T \begin{bmatrix} \frac{1}{s-\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s-\lambda_n} \end{bmatrix} T^{-1}$$

$$= T \operatorname{diag}\left(\frac{1}{s-\lambda_1}, \ldots, \frac{1}{s-\lambda_n}\right) T^{-1}$$

Solution via Diagonalization (1.3)

$$A^k = (T\Lambda T^{-1})^k = (T\Lambda T^{-1})(T\Lambda T^{-1})\cdots(T\Lambda T^{-1})$$

$$= T\Lambda \underbrace{T^{-1}T}\Lambda \underbrace{T^{-1}T}\Lambda T^{-1}\cdots\Lambda T^{-1}$$

$$= T\Lambda^k T^{-1}$$

$$= T\begin{bmatrix}\lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k\end{bmatrix}T^{-1}$$

if $k<0$, this only works if A is invertible, all $\underline{\lambda_i \neq 0}$.

Solution via Diagonalization (2.3)

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$A = T\Lambda T^{-1}$$

$$I = TT^{-1}$$

$$= TT^{-1} + T\Lambda T^{-1} + \frac{T\Lambda^2 T^{-1}}{2!} + \frac{T\Lambda^3 T^{-1}}{3!} + \cdots$$

$$= T\left[I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots\right]T^{-1} = Te^{\Lambda}T^{-1}$$

$$e^{A} = T\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n}\end{bmatrix}T^{-1}$$

Solution via Diagonalization (3.3)

Any function which is analytic (power series)

$$f(A) = T \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} T^{-1}$$

Spectral mapping theorem

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \dots \beta_n A^n$$

$$f: \mathbb{R} \to \mathbb{R}^n$$

$$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \dots + \beta_n a^n$$

Stability of Discrete-Time Systems (1.3)

for what $x(0) \rightarrow x(t) = 0$ as $t \rightarrow \infty$

$\dot{x} = Ax \quad x(0) = \{0\}$ — $x(t) = 0 \;\forall t$.

$\mathrm{Real}(\lambda_1) < 0 \cdots \mathrm{Real}(\lambda_s) < 0 \leftarrow$ STABLE EIGENVALUES

$[v_1 \cdots v_s] \leftarrow$ stable eigenvectors

$\mathrm{Real}(\lambda_{s+1}) \geq 0 \cdots \mathrm{Real}(\lambda_n) \geq 0 \leftarrow$ UNSTABLE EIGENVALUES

$[v_{s+1} \cdots v_n] \leftarrow$ unstable eigenvectors

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} \underbrace{(w_i^T x(0))} v_i$$

$x(0) \in \text{span}\{v_1 \ldots v_s\}$ ← STABLE SUBSPACE.

$\underline{w_i^T x(0) = 0} \ \forall \ i = s+1 \ldots n$

Stability of Discrete-Time Systems (2.3)

suppose A is diagonalizable $\longrightarrow$ $x_{k+1} = Ax_k$

$A = T\Lambda T^{-1}$ $\quad$ $A^k = T\Lambda^k T^{-1}$

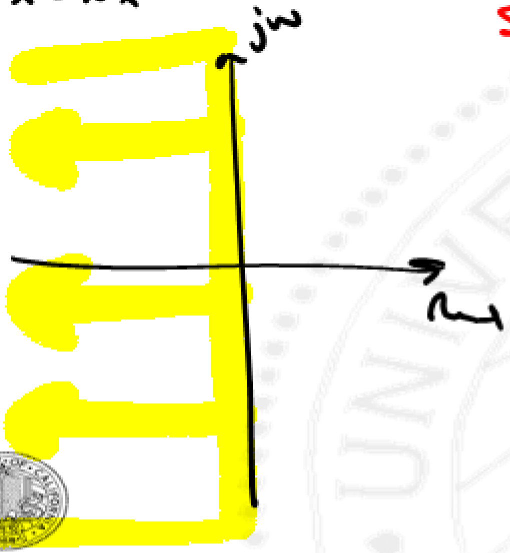
$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

$x_k = A^k x(0) = \sum_{i=1}^{n} \lambda_i^k (w_i^T x(0)) v_i$

$x_k \to 0$ as $k \to \infty$ $\quad$ $|\lambda_i| < 1 \ \forall i$

Stability of Discrete-Time Systems (3.3)

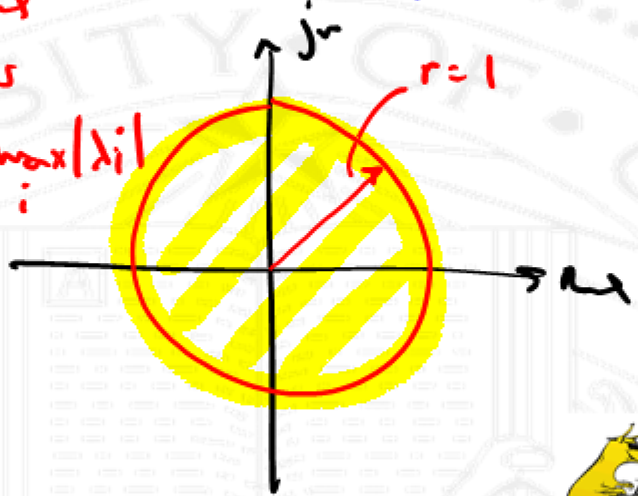
$\dot{x} = Ax$



$x_{k+1} = Ax_k$

Spectral radius

$\rho(A) \triangleq \max_i |\lambda_i|$



Questions?