

CMPE 240: Introduction to Linear Dynamical Systems (or Linear Operator Theory)

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CMPE 240 – Intro. to Linear Dynamical Systems

Overview

- Course Mechanics
- Outline and Topics
- What is a Linear Dynamical System (LDS)
- Why Study Linear Systems
- Some examples



Course Mechanics

- All class info, lectures, homeworks, announcements, etc., on the class webpage at:
 - <https://classes.soe.ucsc.edu/cmpe240/Spring17/>
- Course Requirements:
- Weekly homeworks (hard)
- Takehome Midterm exam (24 hours, TBD)
- Takehome Final exam (24 hours, TBD)



Prerequisites

- Exposure to Linear Algebra (AMS 27)
- Exposure to Differential Equations and Laplace Transforms
- Not required but useful:
 - Control Systems (CMPE 141/EE 154)
 - Circuit Analysis (EE70)
 - Dynamics and Statics (Physics 5/6)
 - Signals and Systems (EE103)



Books

- Not required for the class, but excellent references:
 - Linear Algebra and its Applications, 3rd Edition, G. Strang (Harcourt, 1988), ISBN: 0155510053 (Abebooks.com ~\$7.14 including shipping)
 - Lecture Notes for Stanford's EE263 Class, by Steve Boyd, 2008/2009 (link on website)



Major Topics and Outline

- Linear Algebra and Applications 3-4 weeks
- Autonomous Linear Dynamical Systems - $\dot{\underline{x}} = \underline{A}\underline{x}$
- Linear Dynamical Systems with Inputs and Outputs $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$ $\underline{y} = \underline{C}\underline{x} + \underline{D}\underline{u}$
- Basic Quadratic Control and Estimation



Linear Dynamical System (1.3)

- Vector differential equation
- Continuous time linear dynamical system

$$\frac{d\vec{x}}{dt} = A(t) \vec{x}(t) + B(t) \vec{u}(t)$$

$n \times 1$ $n \times n$ $n \times 1$ $n \times m$ $m \times 1$

$$\vec{x} \in \mathbb{R}^n$$

$$\vec{y}(t) = C(t) \vec{x}(t) + D(t) \vec{u}(t)$$

$p \times 1$ $p \times n$ $p \times m$ $m \times 1$

output/sensors \rightarrow input/control $\in \mathbb{R}^m$

$t \rightarrow$ time $\in \mathbb{R}$



Linear Dynamical System (2.3)

$\vec{x}(t) \in \mathbb{R}^{n \times 1}$ — "state" $\vec{u}(t) \in \mathbb{R}^{m \times 1}$ — input/control

$A(t) \in \mathbb{R}^{n \times n}$ "Dynamics Matrix"

$\vec{y}(t) \in \mathbb{R}^{p \times 1}$
output/sensors

$B(t) \in \mathbb{R}^{n \times m}$ "Input Matrix"

$C(t) \in \mathbb{R}^{p \times n}$ "Output/Sensor Matrix"

$D(t) \in \mathbb{R}^{p \times m}$ "Direct Feedthrough" Matrix

$$\begin{bmatrix} \dot{\vec{x}} = A\vec{x} + B\vec{u} \\ \vec{y} = C\vec{x} + D\vec{u} \end{bmatrix} \rightarrow \begin{bmatrix} \dot{\vec{x}} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{u} \end{bmatrix}$$



Linear Dynamical System (3.3)

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

WEST COAST
STANFORD NOTATION
1950's

$$\begin{aligned}\dot{x} &= Fx + Gu \\ y &= Hx + Ju\end{aligned}$$

EAST COAST
MIT/WHARF
1960's



Some LDS Terminology

Most linear systems are linear Time Invariant (LTI)

$$A, B, C, D \neq f(t)$$

$$\dot{x} = Ax \leftarrow \text{autonomous L.D.S.}$$

very often $D = \emptyset \leftarrow \emptyset^{p \times m}$

if $\bar{u}(t)$ and $\bar{y}(t)$ are both scalars ($m=1, p=1$)

SISO — single input single output

MIMO — multiple input, multiple output ($m \neq 1, p \neq 1$)



Discrete-time LDS

$$\underline{x} : \mathbb{R} \rightarrow \mathbb{R}^n$$

$$\vec{x}(1.35)$$

$$\begin{cases} \dot{\vec{x}} = A\vec{x} + B\vec{u} \\ \vec{y} = C\vec{x} + D\vec{u} \end{cases} \quad n^{\text{th}} \text{ order differential equation}$$

$$\vec{x}(t+1) = A(t) \vec{x}(t) + B(t) \vec{u}(t)$$

$$\vec{y}(t) = C(t) \vec{x}(t) + D(t) \vec{u}(t)$$

$$t \in \mathbb{Z} \{0, \pm 1, \pm 2, \dots\}$$

$$\underline{x} : \mathbb{Z} \rightarrow \mathbb{R}^n$$

$$\begin{cases} \vec{x}_{k+1} = \Phi_k \vec{x}_k + \Gamma_k \vec{u}_k \\ \vec{y}_k = H_k \vec{x}_k + D_k \vec{u}_k \end{cases} \quad n^{\text{th}} \text{ order recursion/difference eq'n.}$$



Why Study Linear Systems

Used to be super-complex, esoteric, not very used.

Applications are everywhere

Mainstream

- Automatic Control Systems
- Signal Processing (Kalman Filter)
- Communications
- Econ/Finance
- Circuit Analysis (SPICE)
- Finite Element Analysis (FEA) - mechanical
- Aerodynamics (CFD)

GPS - Navigation

Machine Learning

Complex Systems



Usefulness of LDS

Depends on our abilities to do compilation

Morrie's law is an overside

1980's $\sim n=100$.

$n=100K$

LAPACK



Origins and History

LEAST SQUARES \rightarrow GAUSS, KOPERNICUS 1600's

LINEAR ALGEBRA \rightarrow 19th century

CLASSICAL OPTIMAL CONTROL \rightarrow 19th century

CONTINUOUS DYNAMICS SYSTEM - 1920's BELL LABS

FIRST REAL APPLICATION \rightarrow 1940's - SOVIET PRICE
CONTROLS

MILITARY / SPACE PROGRAMS



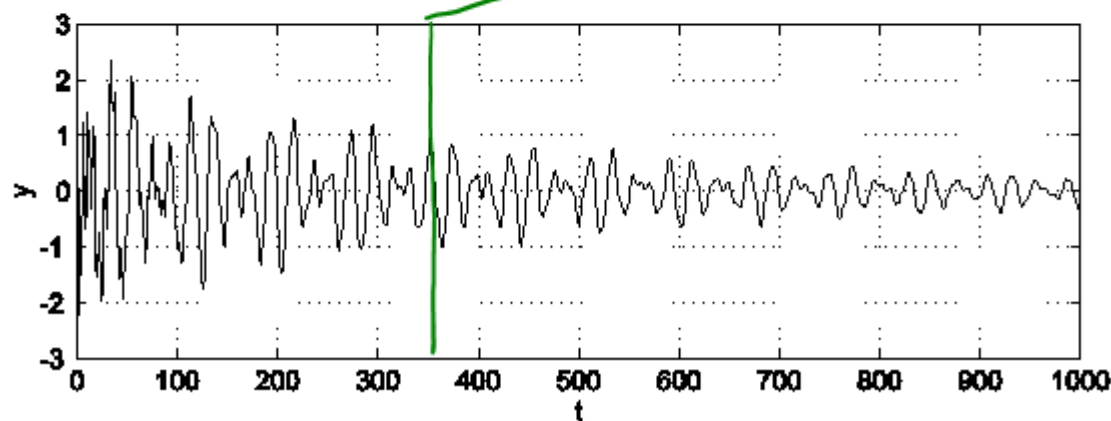
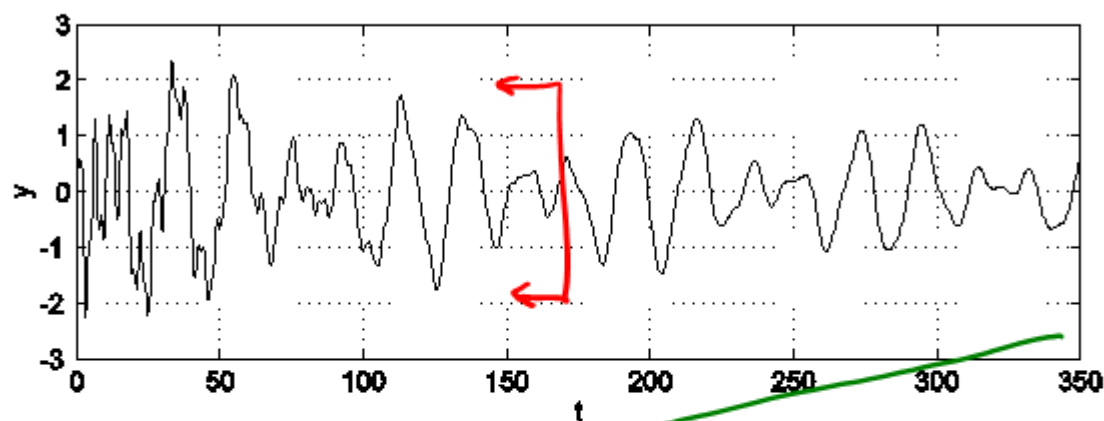
Non-linear Dynamical Systems

- most systems are, in fact, non-linear
- most techniques for solving non-linear systems are based on linear solutions
- If you don't understand the linear stuff, you'll never understand non-linear.

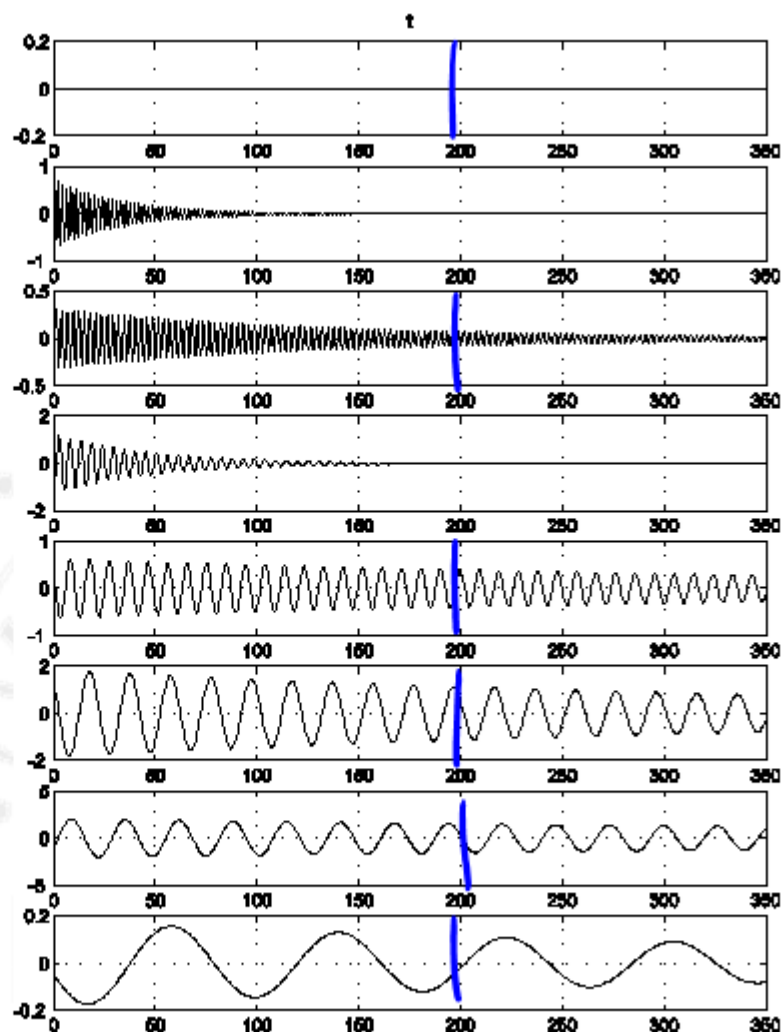


$h=16$

Examples – Mechanical



Spectral
Leakage



Questions

Comment → CANVAS Homework Turn in is up.

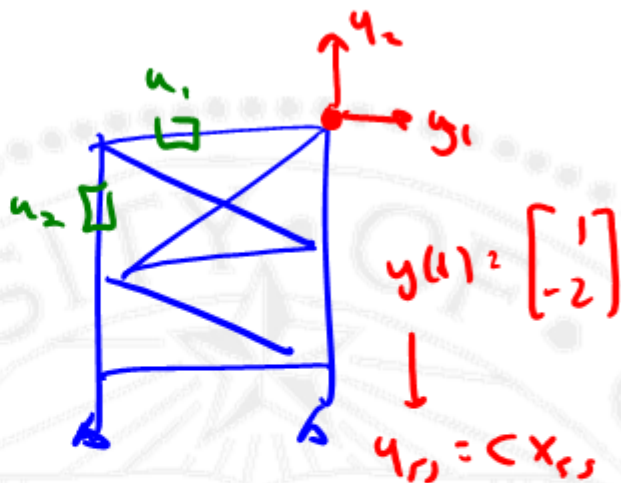


Input Design – Mechanical

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$u: \mathbb{R}_+ \rightarrow \mathbb{R}^n$$



$$0 = Ax + Bu_{ss}$$

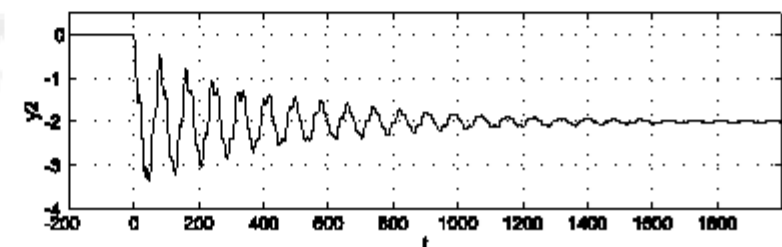
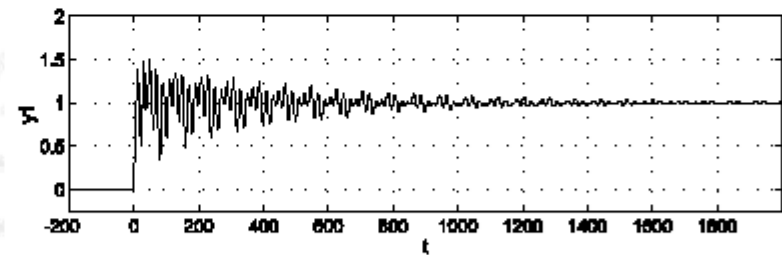
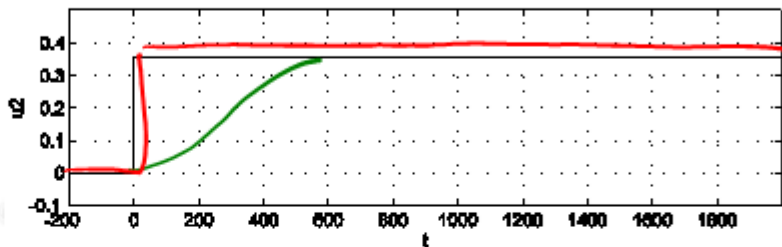
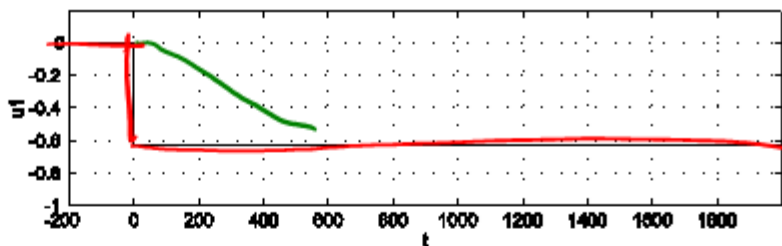
$$-\tilde{A}B u_{ss} = \tilde{A}x_{ss}$$

$$-\tilde{A}B u_{ss} = x_{ss} \quad y_{ss} = -C\tilde{A}B u_{ss}$$

$$u_{ss} = -[C\tilde{A}B]^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$$

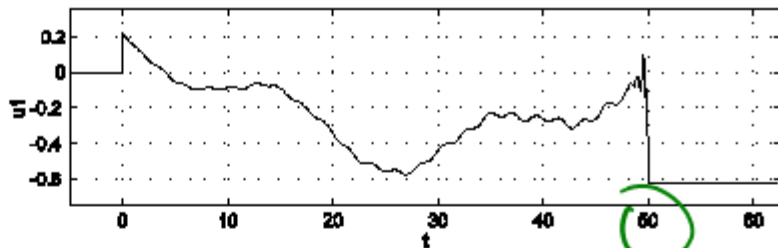


$$u_{ss} = \begin{bmatrix} -0.63 \\ +0.37 \end{bmatrix}$$

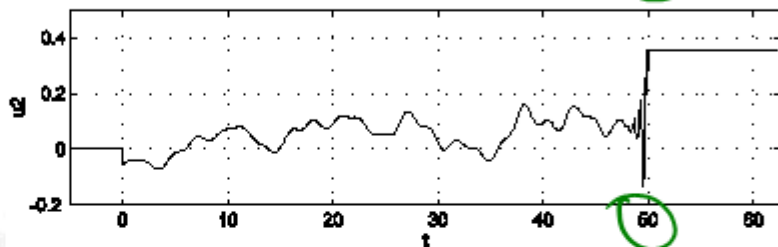


VERST
NOTES

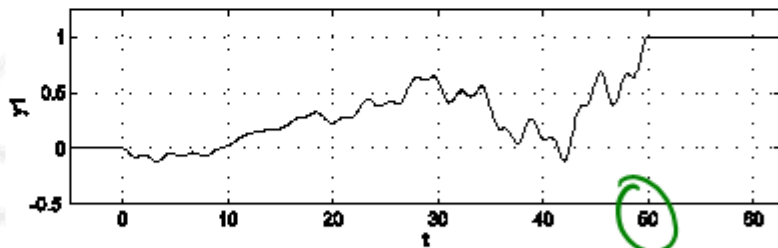
u_1



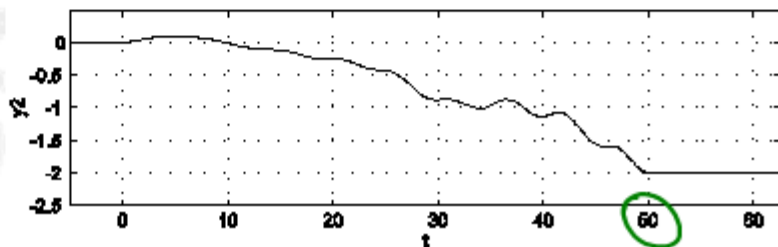
u_2

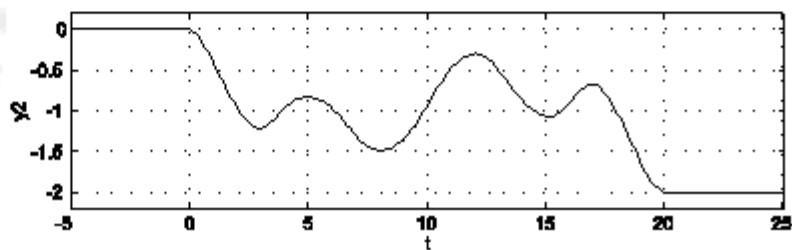
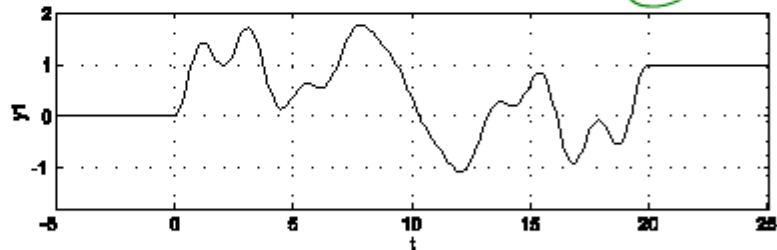
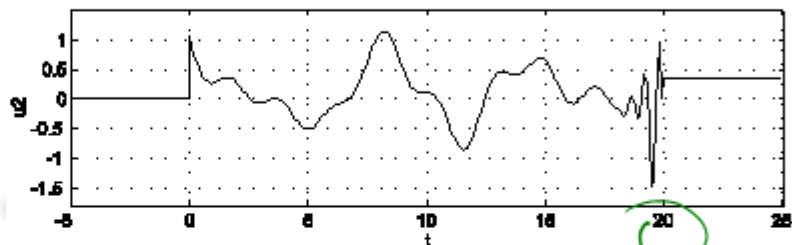
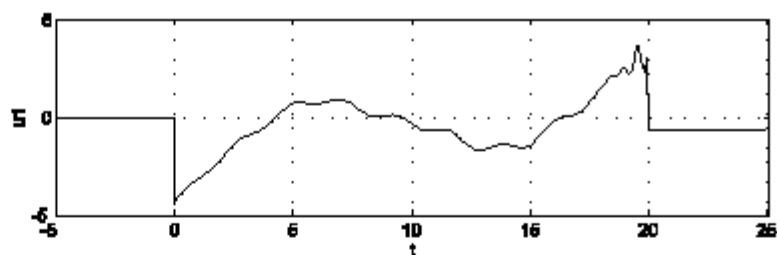
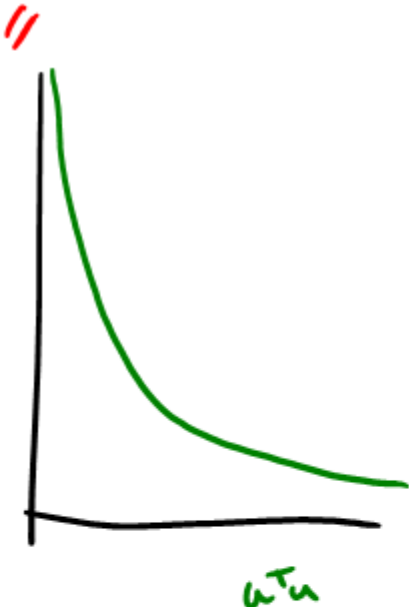


y_1

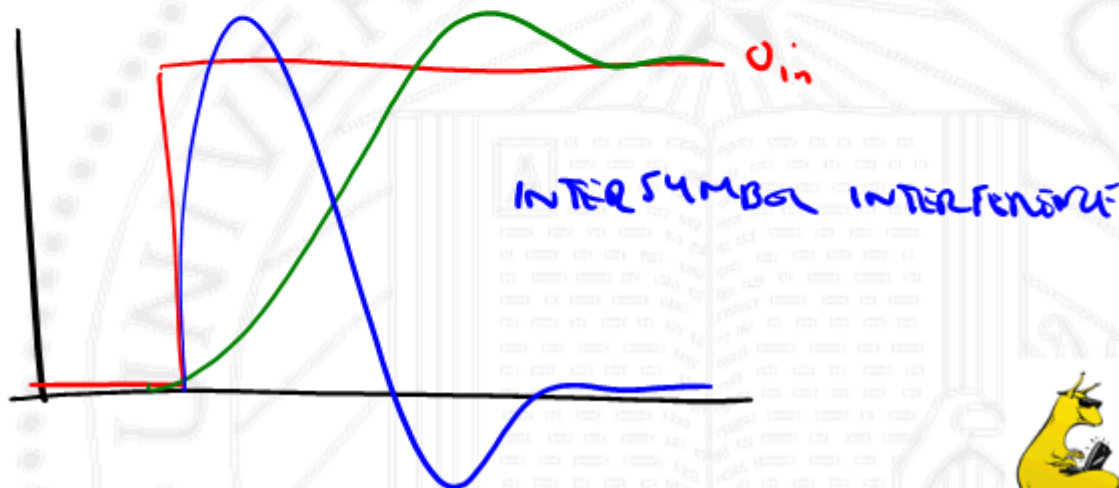
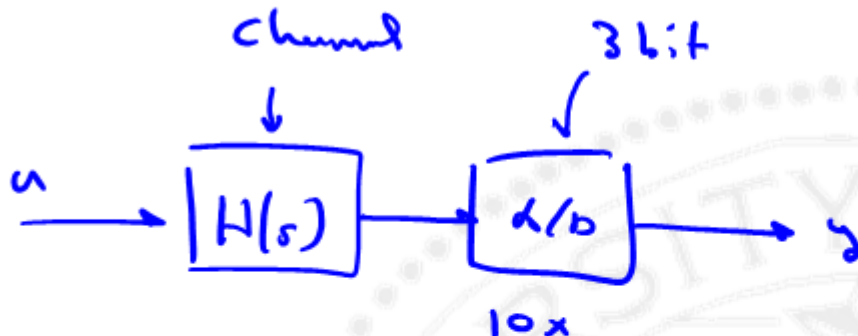


y_2

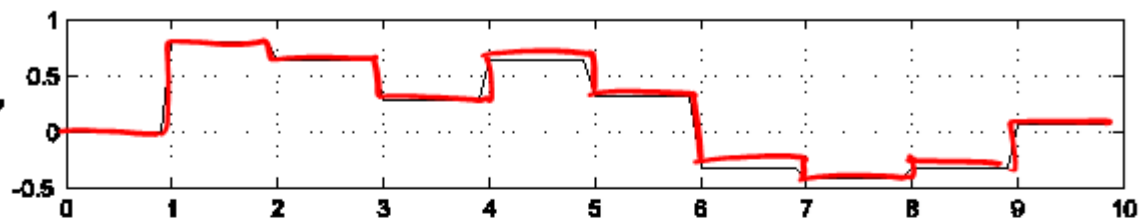




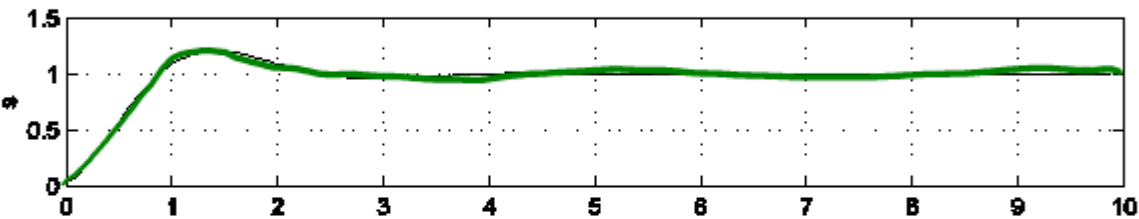
Estimation / Filtering



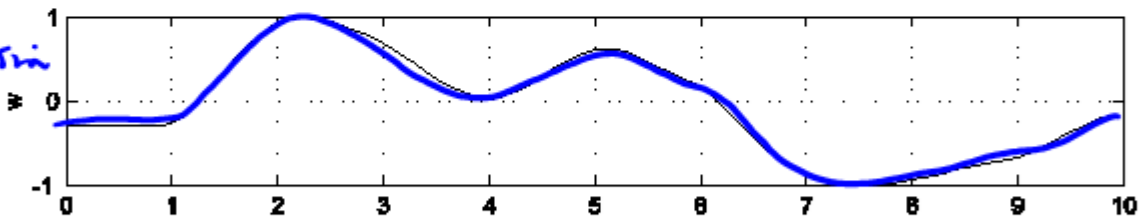
$u(t)$



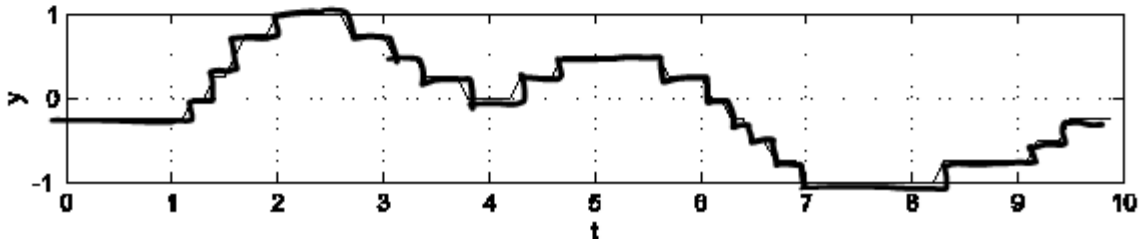
$H(s)$



convolution

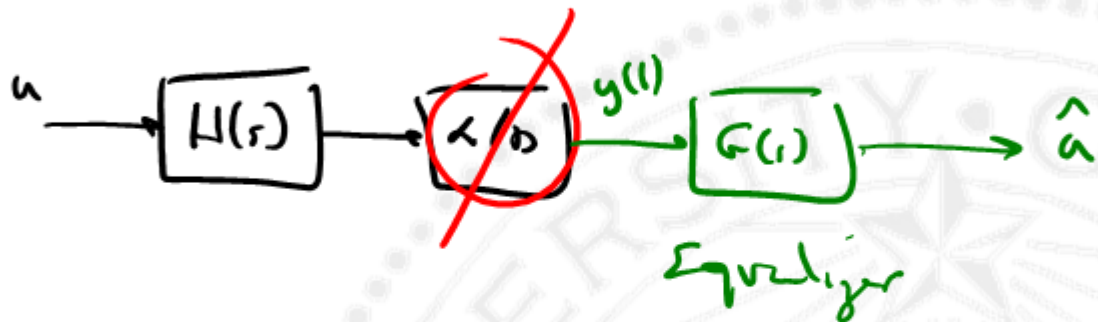


$q(t)$



Simple Approach

1. ~~noise~~



$$\underline{G(s)H(s) \approx 1}$$

$$\underline{0.25V}$$

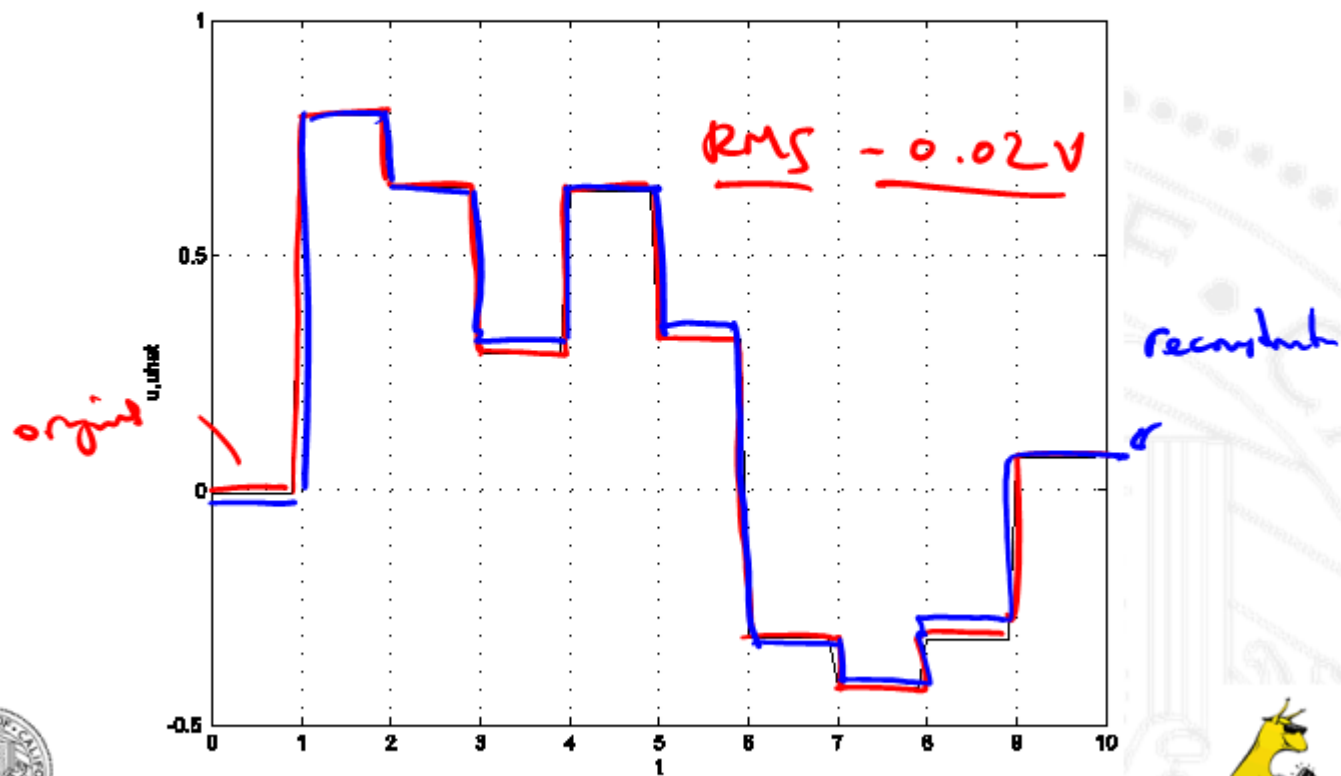
$$\pm \underline{0.125V}$$

WORKS TERRIBLY

RMS ERROR $\approx 2\sigma$



Estimation Approach

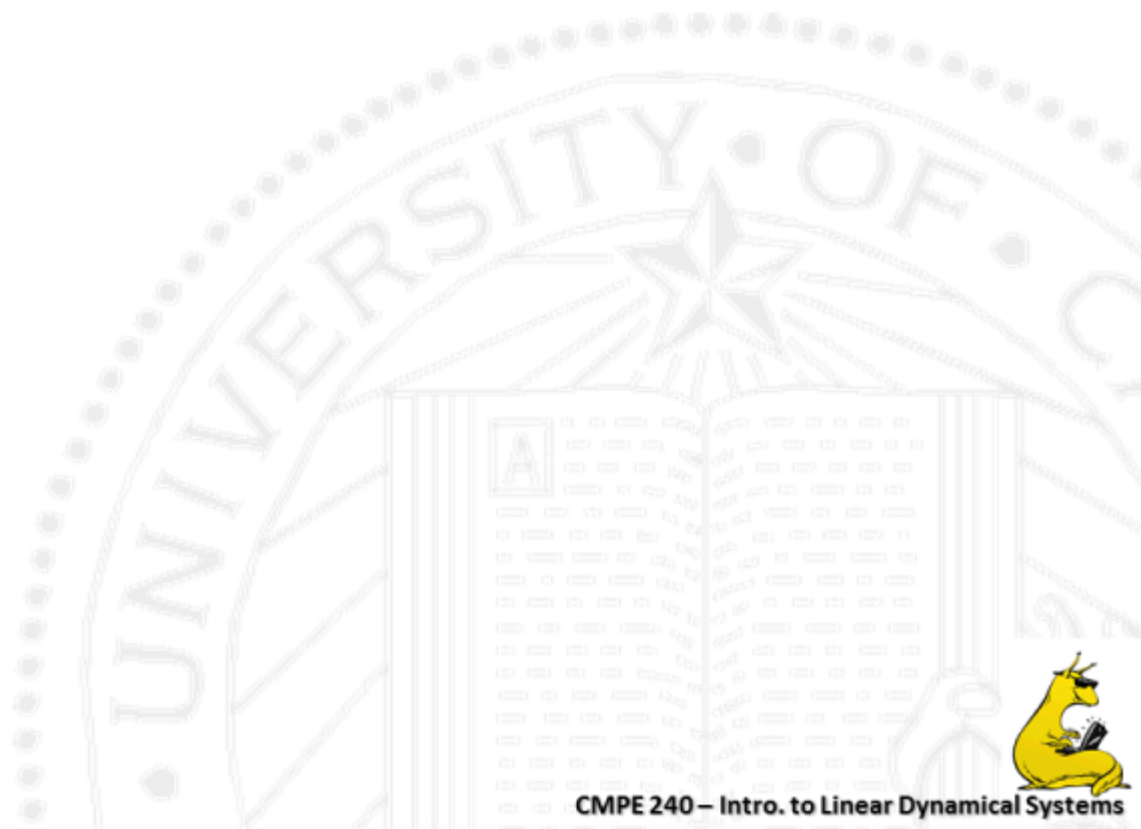


Questions?





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