Solution via Laplace Transform and Matrix Exponential

Gabriel Hugh Elkaim

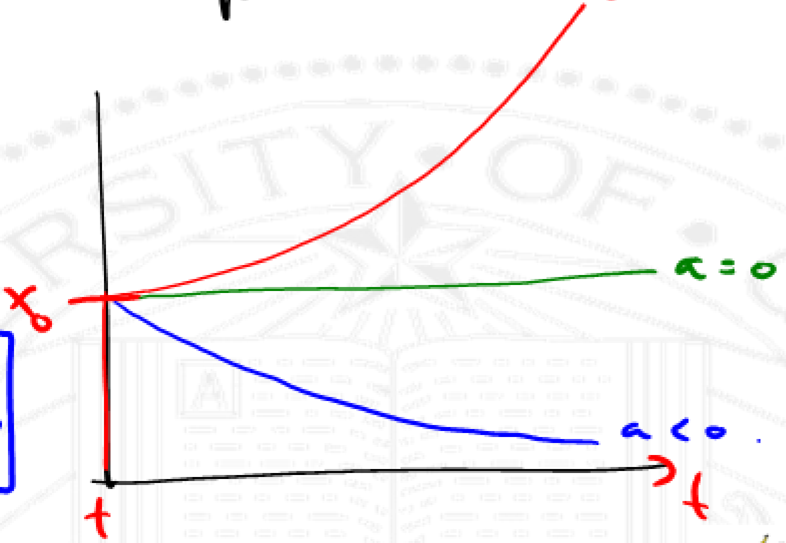
Midterm out @ end of class.

Solution via Laplace Transform and Matrix Exponential

- Laplace Tranform
- Solving x’ = Ax via Laplace Tranform
- State transition matrix
- Matrix Exponential
- Qualitative behavior and Stability

$\dot{x} = ax$ ← scalar equation

$$\boxed{x(t) = x_0 e^{at}}$$



$\underline{\dot{x}} = A\underline{x} \rightarrow \boxed{\underline{x}(t) = e^{At}\underline{x}_0}$

MATRIX EXPONENTIAL

Laplace Transform of a Matrix (1.3)

$$\mathcal{L}\{x(t)\} \triangleq \int_0^\infty x(t)\, e^{-st}\, dt \triangleq X(s)$$

$x: \mathbb{R}_+ \to \mathbb{R}$

$X: \mathbb{C} \to \mathbb{C}$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$$

$$\mathcal{L}\{\dot{x} = ax\} = sX(s) - x_0 = aX(s) \longrightarrow (s-a)X(s) = x_0$$

$$\boxed{X(s) = \frac{x_0}{s-a}}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0}{s-a}\right\} = \underline{x_0 e^{at}}$$

Laplace Transform of a Matrix (2.3)

$g : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$ ← integrating, matrix valued signal

$Z = \mathcal{L}\{g\}$ where $Z : D \subseteq \mathbb{C} \to \mathbb{C}^{p\times q}$

complex

$$Z(s) = \int_0^\infty e^{-st} g(t)\, dt$$

$D \triangleq$ domain of convergence

includes a half-plane at least $\{s \mid \text{real}(s) > a\}$

where a: $|g_{ij}(t)| \leq \alpha e^{at}$ $t > 0,\ i=1..p,\ j=1..q$

Laplace Transform of a Matrix (3.3)

$$\mathcal{L}\{\overbrace{z(t)}^{\mathbb{R}^{p\times q}}\} = \overbrace{Z(s)}^{\mathbb{C}^{p\times q}}$$

$\mathbb{R}_+$ $\mathbb{C}$

lower case (t) ← time

upper case (s) ← freq.

Derivative Property (1.2)

$$\mathcal{L}\{\dot{g}\} = sG(s) - g_0$$

$$\int u\,dv = uv - \int v\,du$$

$$\mathcal{L}\{\dot{g}\} = \int_0^\infty \underbrace{\dot{g}(t)}_{dv}\,\underbrace{e^{-st}}_{u}\,dt = e^{-st} g(t)\Big|_{t=0}^{\infty} + \int_0^\infty g(t)e^{-st}(-s\,dt)$$

$$= 0 - 1\,g(0) + s\underbrace{\int_0^\infty g(t)e^{-st}\,dt}_{G(s)}$$

Derivative Property (2.2)

$\mathcal{L}\{\ddot{z}\} = s^2 Z(s) - s z_0 - \dot{z}_0$

$\mathcal{L}\{\dot{x} = Ax\} = sX(s) - x_0 = AX(s) \longrightarrow sX(s) - AX(s) = x_0$

$[sI - A]X(s) = x_0 \longrightarrow \boxed{X(s) = [sI - A]^{-1} x_0}$

$x(t) = \mathcal{L}^{-1}\{[sI - A]^{-1} x_0\}$

Laplace transform solution of x=Ax (1.3)

$A \in \mathbb{R}^{n \times n}$ $\qquad x \in \mathbb{R}^n$

$X(s)$ complex $n \times n$ function

$(sI - A)^{-1} \triangleq$ Resolvent of A (1850)

$s \in \mathbb{C}$ except for eigenvals of A.

Laplace transform solution of $\dot{x} = Ax$ (1.3)

eigenvalues of A : $\det(sI - A) = 0$.

$X(s) = [sI - A]^{-1} x_0$

$x(t) = \mathcal{L}^{-1}\{[sI - A]^{-1} x_0\}$

$x(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}\} x_0$

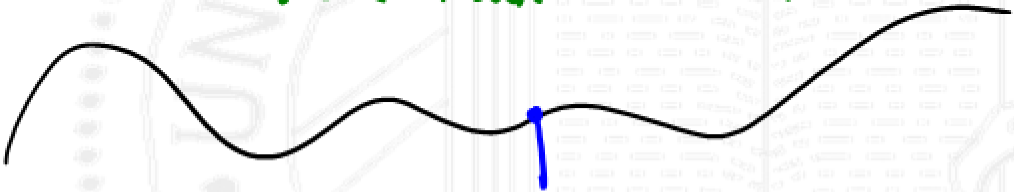
Laplace transform solution of

$$\dot{x} = Ax \quad (2.3)$$

$$x(t) = \underbrace{\mathcal{L}^{-1}\{[sI-A]^{-1}\}}_{\Phi(t) \sim \mathbb{R}^{n\times n}} x_0$$

$$x(t) = \Phi(t)\, x_0 .$$

state transition matrix,



Laplace transform solution of $\dot{x} = Ax$ (3.3)

$$\underbrace{x(t)}_{n\times 1} = \underbrace{\begin{bmatrix} \phi \end{bmatrix}}_{\Phi} \underbrace{x_0}_{n\times 1}$$

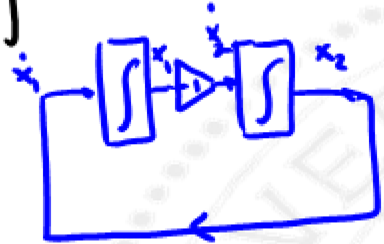
$\Phi(t) \to 0$ as $t \to \infty$

"STABILITY"

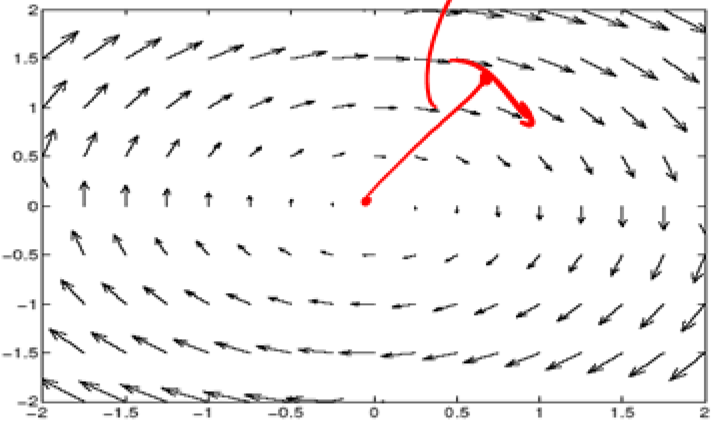
Autonomous LDS example (1.3)
Harmonic Oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$



||x|| const

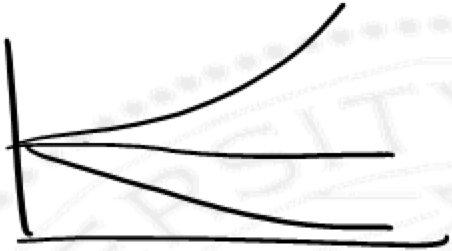


$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \longrightarrow \mathcal{L}^{-1}\{\} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Autonomous LDS example (2.3)
Harmonic Oscillator

$\dot{x} = a x$

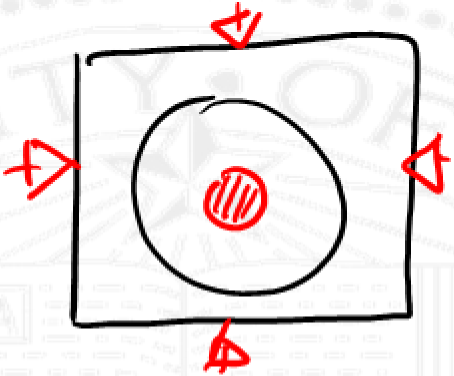


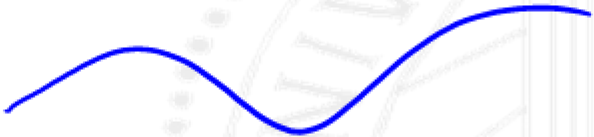
$\dot{x} = A x \longrightarrow x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x_0$

Autonomous LDS example (3.3)
Harmonic Oscillator

Gravity Probe B

Drag-free



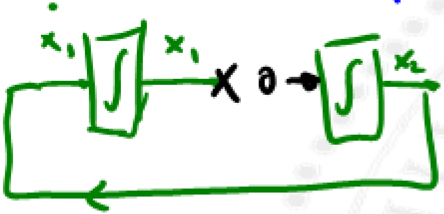


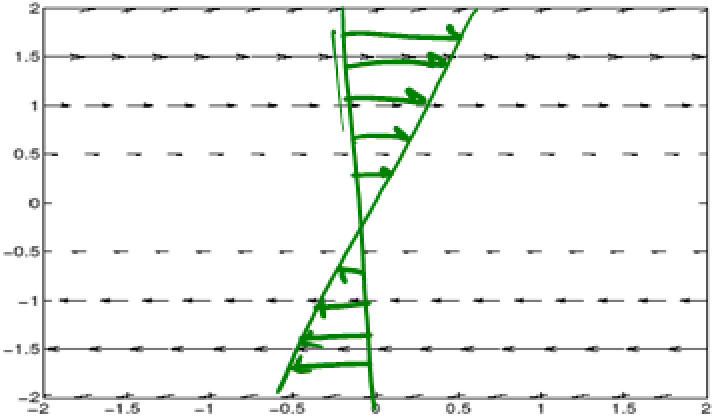
Autonomous LDS example (1.3)
Double Integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

$$\dot{x}_2 = 0$$

$$\dot{x}_1 = x_2$$





$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \longrightarrow \Phi = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Autonomous LDS example (2.3)
Double Integrator

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x_0$$

$$\mathrm{eig}(sI - A) = (0, 0)$$

$$\det(sI - A) = 0 \text{ when } s = 0.$$

$$\frac{1}{s^2}.$$

Autonomous LDS example (3.3)
Double Integrator

Characteristic Polynomial (1.2)

$\chi(s) \triangleq \det(sI - A)$ characteristic polynomial of A, $[\Delta(s)]$.

$\chi(s)$ is a polynomial of degree n

(monic polynomial)

$$s^n + (\Sigma a_{ii}) s^{n-1} + \dots + \Pi a_{ii}$$

$\det \begin{bmatrix} s-a_{11} & a_{12} & \cdots \\ a_{21} & & \end{bmatrix}$

//

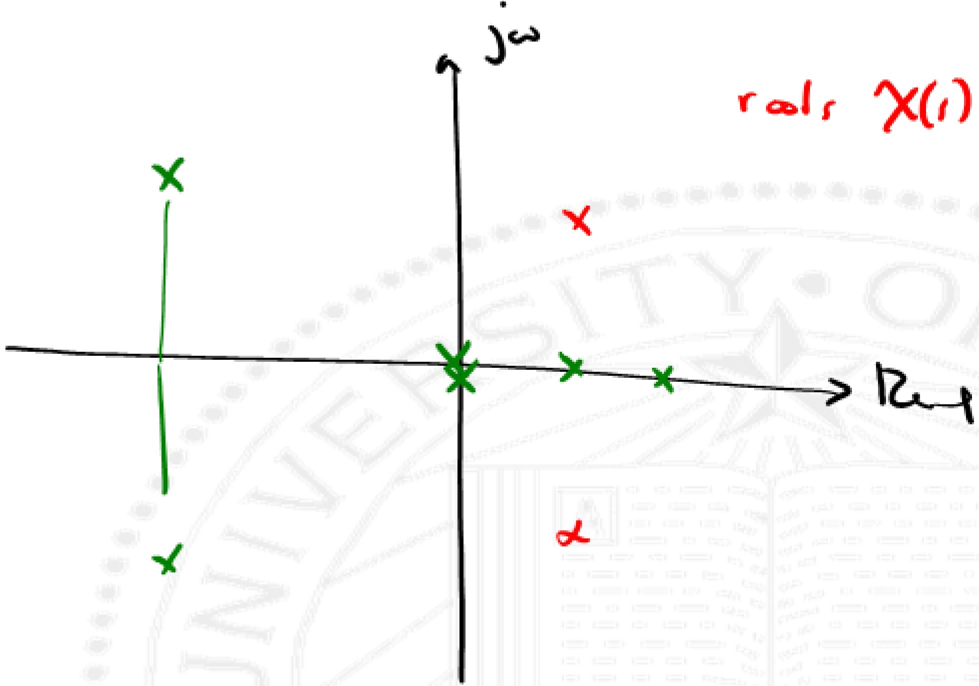
Characteristic Polynomial (2.2)

$\mathcal{X}(s)$ has n eigenvalues (if we count multiplicity of roots)

roots of $\mathcal{X}(s)$ are the eigenvalues of A.

$\mathcal{X}(s)$ has only REAL coefficients. Eigenvalues are either REAL or they come in COMPLEX CONJUGATE pairs

$(\sigma \pm j\omega)$


jω
α

Eigenvalues of A and Poles of Resolvent (1.3)

$[sI - A]^{-1}$ $\longrightarrow$ (i,j) entry of the resolvent

Cramer's rule for matrices $[sI-A]^{-1} \triangleq \dfrac{\text{ADJ}[sI-A]}{\text{DET}[sI-A]}$

$$(-1)^{i+j}\frac{\det \Delta_{ij}}{\det(sI-A)}$$

$\det \Delta_{ij}$ $\leftarrow$ dgree $< n$ $(n-1)$

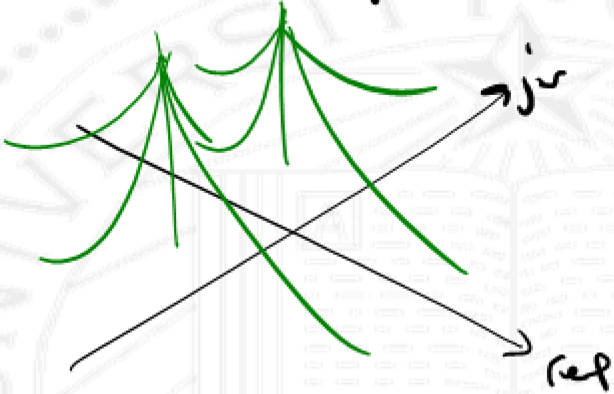
$\det(sI-A)$ $\longleftarrow$ $\chi(s)$ dgree n

$\Delta_{ij} = sI - A$ with the j^{th} row & i^{th} column deleted

rational functions in "s"

Eigenvalues of A and Poles of Resolvent (2.3)

$\det[sI - A] = \emptyset$ ← blows up resolvent



Eigenvalues of A and Poles of Resolvent (3.3)

The Matrix Exponential (1.3)

$\frac{1}{1-c} = 1 + c + c^2 + c^3 + \ldots$ works if $|c| < 1$

$[I - c]^{-1} = I + c + c^2 + c^3 + \ldots$ if series converges

$[sI - A]^{-1} = \frac{1}{s}\left[I - \frac{A}{s}\right]^{-1} = \frac{1}{s}\left[I + \frac{A}{s} + \frac{A^2}{s^2} + \frac{A^3}{s^3} + \ldots\right]$

$= \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots$ converges as long as "s" is big enough.

The Matrix Exponential (2.3)

$$\Phi \triangleq \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \mathcal{L}^{-1}\left\{\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \dots\right\}$$

$$e^{at} \triangleq 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \dots$$

$$\Phi = \mathcal{L}^{-1}\left\{\frac{I}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s^2}\right\} + \dots$$

MATRIX EXPONENTIAL

$$\Phi = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots = e^{At}$$

The Matrix Exponential (3.3)

$$e^{M} \triangleq I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots \qquad M \in \mathbb{R}^{n\times n}$$

$$\Phi(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = e^{At}$$

$$\boxed{x(t) = e^{At} x_0}$$ ← non-trivial overloading.

$$\exp(A) = \begin{bmatrix} e^{a_{11}} & e^{a_{12}} & \dots \\ \vdots & & \end{bmatrix}$$

$\underline{\text{expm}}(A)$ ← matrix exponential

Matrix Exponential Solution of A-LDS (1.5)

$e^a e^b = e^{(a+b)}$ ← scalar world $ab = ba$

$ab = 0$ ← a, or b is 0.

$e^A e^B \neq e^{A+B}$ ← matrix world $AB \neq BA$

$AB = 0$ does not mean A or $B = 0$.

Matrix Exponential Solution of A-LDS (2.5)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A+B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$e^{A} = \begin{bmatrix} 0.59 & 0.89 \\ -0.89 & 0.59 \end{bmatrix} \qquad e^{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Matrix Exponential Solution of A-LDS (3.5)

$$e^{A}e^{B} = \begin{bmatrix} 0.54 & 1.38 \\ -0.54 & -0.30 \end{bmatrix}$$

$$\neq$$

$$e^{(A+B)} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix}$$

$$\boxed{e^{A+B} = e^{A}e^{B} \text{ iff } AB = BA}$$

Matrix Exponential Solution of A-LDS (4.5)

$t, s \in \mathbb{R}$ $\quad e^{At} e^{As} = e^{A(t+s)}$

$At\,As = ts\,A^2$

$As\,At = ts\,A^2$

$t = 1,\ s = -1$

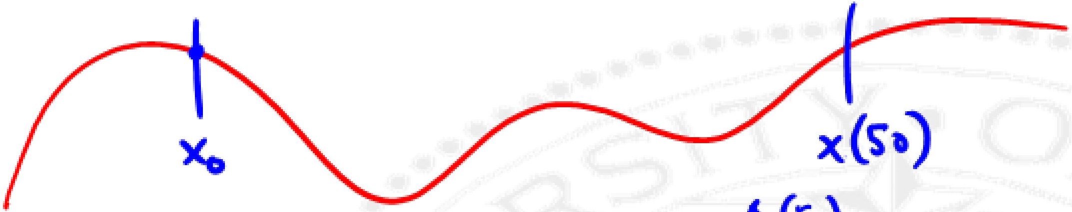
$e^{A(-1)} e^{A(1)} = e^0 = I$

$$\boxed{\left[e^{At}\right]^{-1} = e^{-At}}$$

e^{At} non-singular

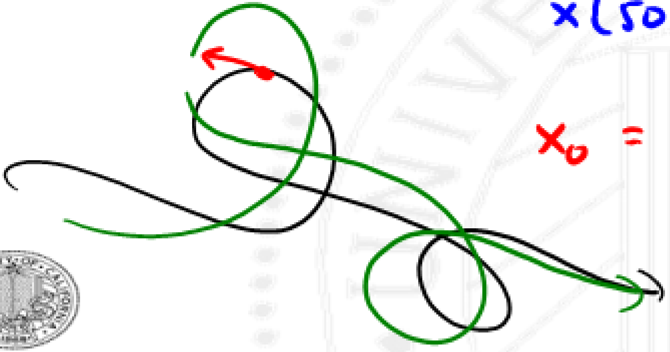
Matrix Exponential Solution of A-LDS (5.5)

$\dot{x} = Ax$



$$x(50) = e^{A(50)} x_0$$

$$x_0 = e^{A(-50)} x(50)$$



Time Transfer Property (1.2)

e^{At}

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \mathcal{L}^{-1}(sI - A)^{-1} \rightarrow \boxed{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \leftarrow e^{At}}$$

$t = 2 \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$e^{At} = \underbrace{I + At} + A^2 \frac{t^2}{2!} + \cdots$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$t = -2 \quad \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Time Transfer Property (2.2)

$\dot{x} = Bx \qquad 0 \le t < 1$

$\dot{x} = Ax \qquad 1 \le t < \infty$

$x(0) = x_0$

$x(t) = e^{A(t-1)} x(1)$

$x(1) = e^{Bt} x_0$

$$x(t) = \begin{cases} e^{Bt} x_0 & 0 \le t < 1 \\ e^{A(t-1)} e^{B} x_0 & t \ge 1 \end{cases}$$

$x(t+\tau) = e^{At} x(\tau)$

$\dot{x} = Ax \longrightarrow x(t) = \Phi(t)\, x_0 = e^{At} x_0$

e^{At} propagates an initial state into a state at time t.

$x(t+\tau) = e^{At} x(\tau)$

Forwards or Backwards in time

$\dot{x} = A x$

$$x(t+\tau) = x(\tau) + t\,\dot{x}(\tau) = [I + At]\,x(\tau)$$

FORWARDS INTEGRATION

$$x(t+\tau) = e^{At}\,x(\tau) = \left[I + At + \frac{A^2 t^2}{2} + \dots\right] x(\tau)$$

$$x(t+\tau) = [I - At]^{-1}\,x(\tau) = \left[I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{4} + \dots\right] x(\tau)$$

BACKWARDS INTEGRATION

Sampling a continuous-time system (1.2)

$\dot{x} = Ax$ sample x @ $t_1 \le t_2 \le t_3 \ldots$ $z_k \doteq x(t_k)$

$$z_{k+1} = e^{A(t_{k+1} - t_k)} z_k$$

$t_{k+1} - t_k = h \quad \forall k.$

$$\boxed{z_{k+1} = e^{Ah} z_k}$$

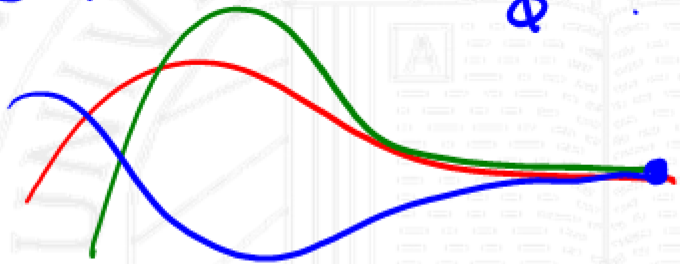
Discrete Version of CDS.

Sampling a continuous-time system (2.2)

$$z_{k+2} = \underbrace{e^{Ah} e^{Ah}}_{(e^{Ah})^2} z_k$$

$$z_k = \Phi^k z_0$$

$$\Phi^{-k}$$



Piecewise Constant System (1.2)

$$A(t) \begin{cases} A_0 & 0 \le t < t_1 \\ A_1 & t_1 \le t < t_2 \\ A_2 & \\ \vdots & \end{cases}$$

jump linear systems

power systems

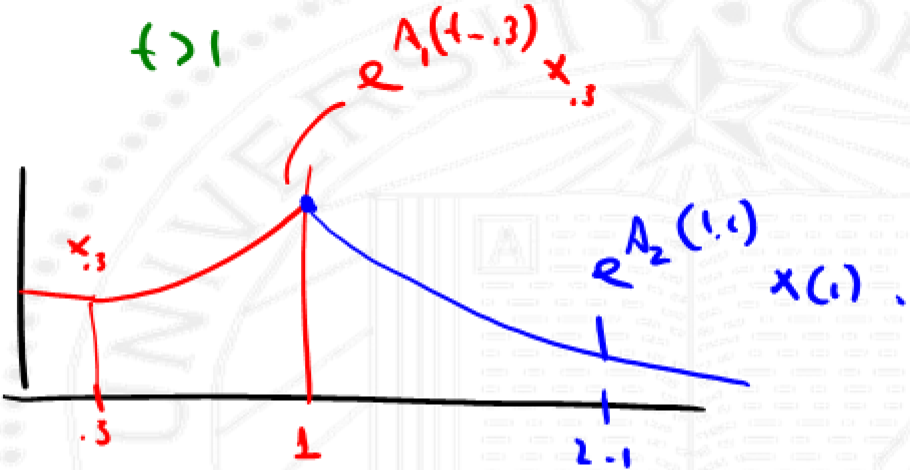
hybrid systems

$$x(t) = e^{A_i(t-t_i)} e^{A_{i-1}(t_i - t_{i-1})} \cdots e^{A t_1} x_0 .$$

Piecewise Constant System (2.2)

$\dot{x} = A_1 x \qquad 0 \le t < 1$

$\dot{x} = A_2 x \qquad t > 1$



Qualitative behavior of *x(t)* (1.3)

$\dot{x} = Ax$

$x(t) \in \mathbb{R}^n$

$x(t) = \underbrace{e^{At}}_{\text{Time}} x_0$

$X(s) = \underbrace{[sI - A]^{-1}}_{\text{Freq.}} x_0$

i^{th} component of $X_i(s) = \dfrac{a_i(s)}{\mathcal{X}(s)} \leftarrow \det(sI - A)$

a_i polynomial in s, degree $< n$.

Qualitative behavior of $x(t)$ (2.3)

Assume we have distinct eigenvalues $\lambda_i \neq \lambda_j$

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

β_{ij} depends linearly on x_0.

$\dot{x} = Ax$ has a response that is a sum of the (complex) frequencies of A.

Qualitative behavior of $x(t)$ (3.3)

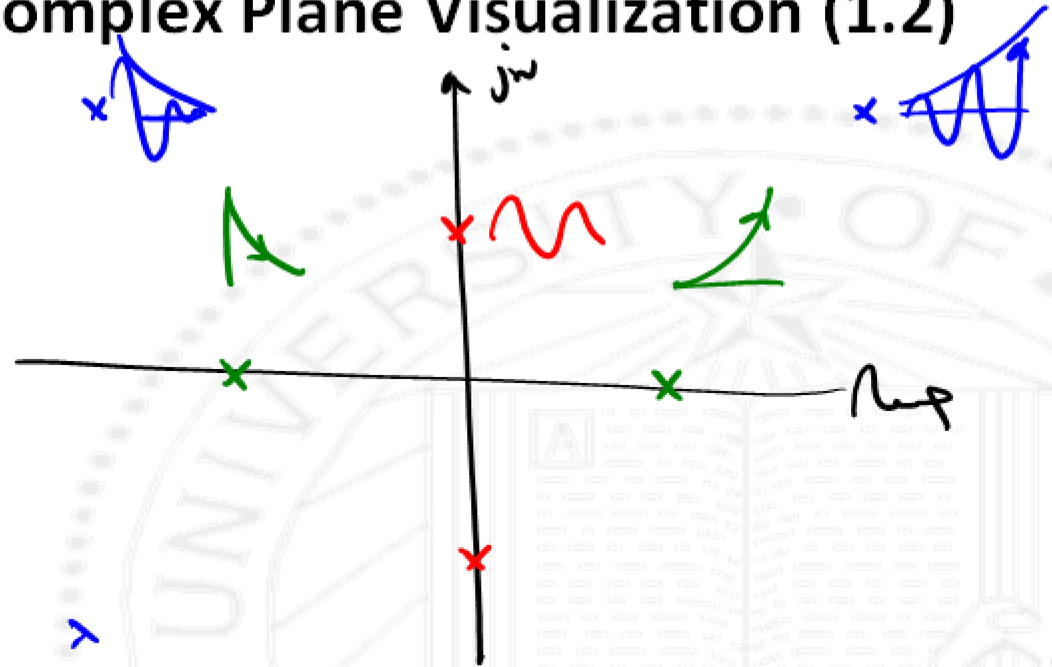
Eigenvalues of A give the exponents that can occur in e^{At}.

Real Eigenvalues: λ corresponds to exponential growth or decay

Complex Eigenvalues: $\lambda = \sigma \pm j\omega$ corresponds to a decaying or growing sinusoid

term $e^{\sigma t} \cos(\omega t + \phi)$

Complex Plane Visualization (1.2)



MIDTERM

1 pm tomorrow PDF.

Piazza — midterm

Points: Post on Piazza

Complex Plane Visualization (2.2)

Suppose A has repeated eigenvalues such that

x_i can have repeated poles

$\lambda_1 \dots \lambda_p$ distinct $n_1 - n_p$ $(\sum n_i = n)$

$$x_i(t) = \sum_{j=1}^{r} P_{ij}(t) e^{\lambda_j t} \qquad t, t^2, t^3 \dots$$

$P_{ij}(t)$ is a polynomial of degree $< n_j$

Stability (1.3)

$\dot{x} = Ax$ "stable" $e^{At} \to 0$ as $t \to \infty$

stable: $x(t) \to \{0\}$ as $t \to \infty$ no matter what x_0.

all trajectories of $\dot{x} = Ax$ converge to $\{0\}$ as $t \to \infty$

$\dot{x} = Ax$ is STABLE iff all eigenvalues of A have negative real parts.

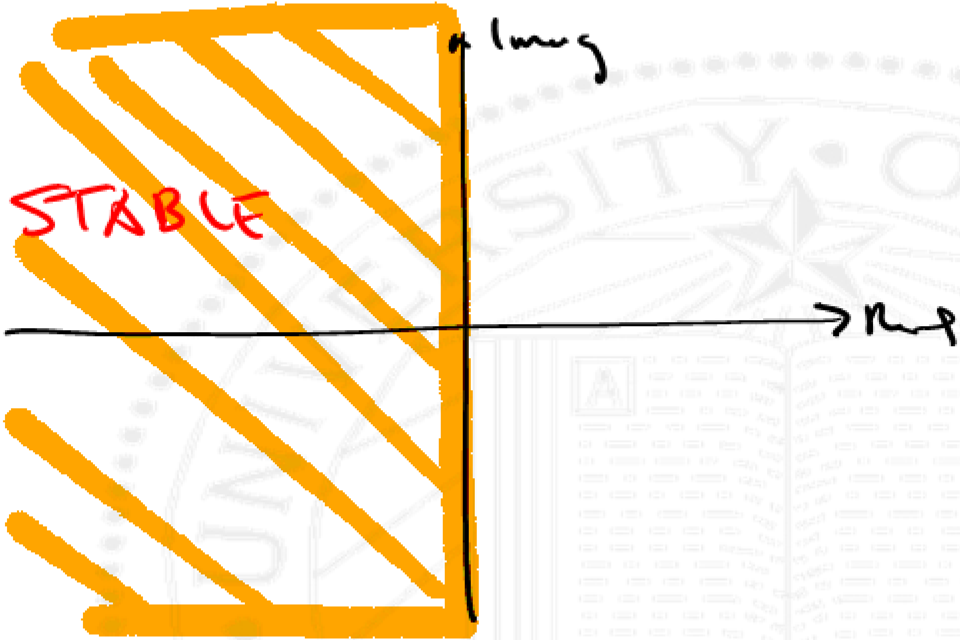
$$\boxed{\mathrm{REAL}(\mathrm{EIG}(A)) = \mathrm{REAL}(\lambda_i) < 0 \quad i = 1 \ldots n}$$

Stability (2.3)

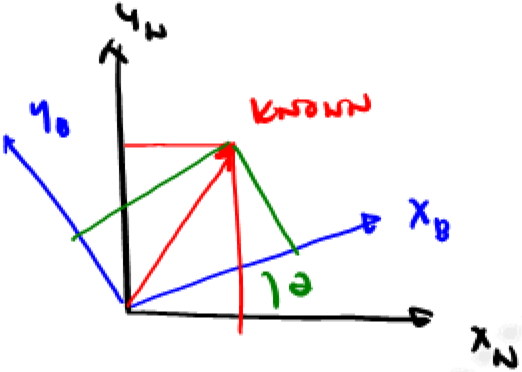
$$\lim_{t \to \infty} p(t) e^{\lambda t} = 0 \text{ for any polynomial } p(t)$$

if $\text{real}(\lambda) < 0$.

Stability (3.3)

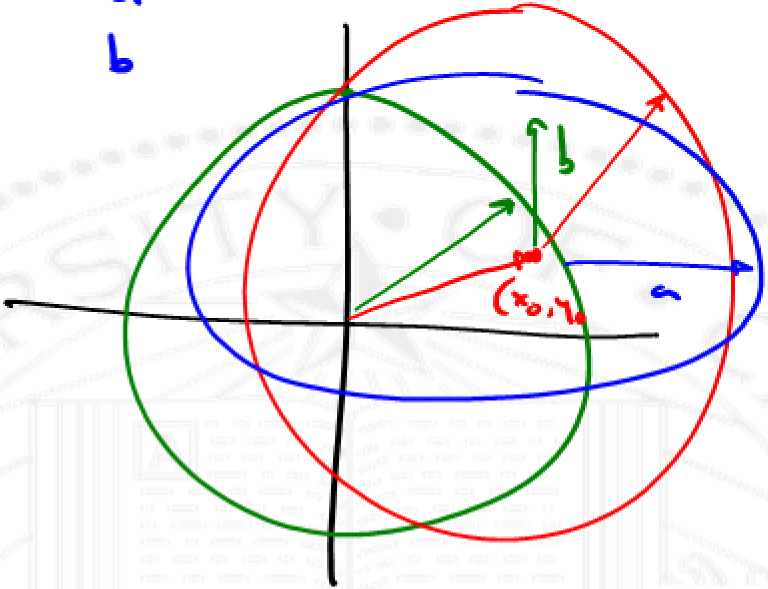


Questions?



x_0 a

y_0 b



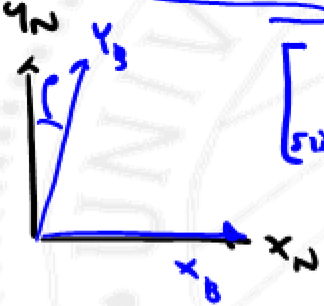
θ is what you want to extract

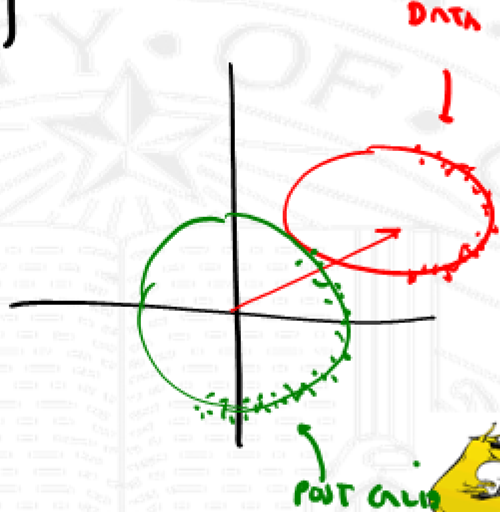
$$\begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_N \\ y_N \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

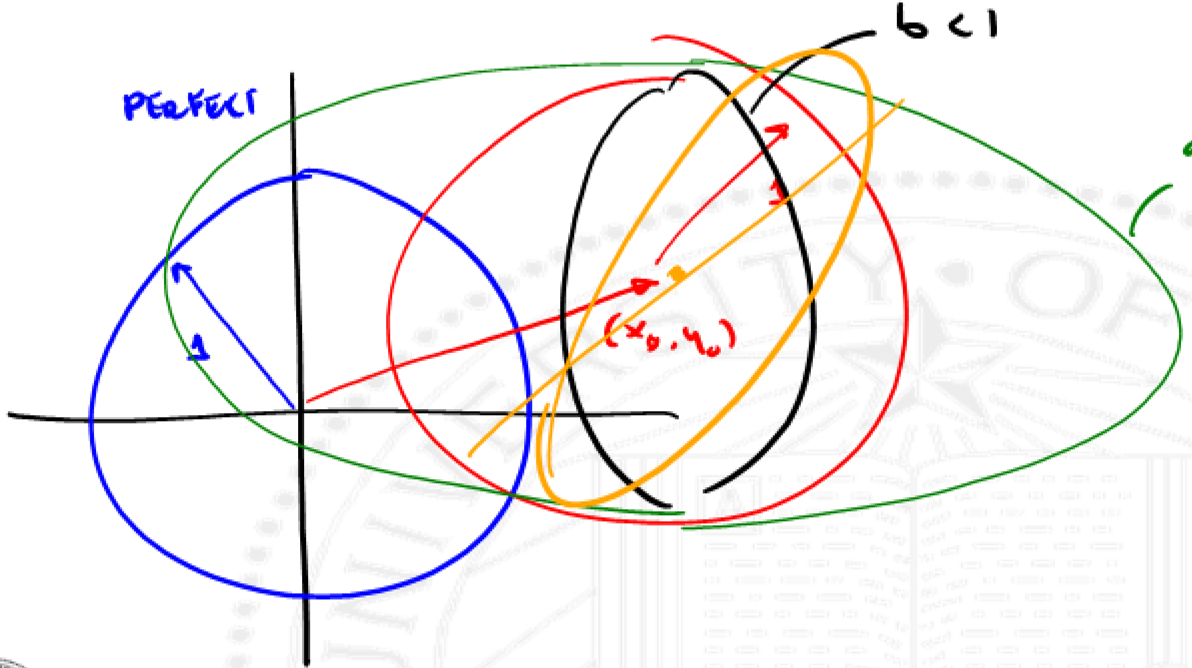
measurement

KNOWN (0,1)

$$\begin{bmatrix} 1 & 0 \\ \sin\rho & \cos\rho \end{bmatrix}$$






PERFECT
1
1
(x_0, y_0)
b < 1
a > 1

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = R^2$$

$$\frac{x^2-2xx_0+x_0^2}{a^2} + \frac{y^2-2yy_0+y_0^2}{b^2} = R^2$$

$$x^2-2xx_0+x_0^2+\frac{a^2}{b^2}y^2-\frac{2}{b^2}yy_0+y_0^2\frac{c^2}{b^2}=cR^2$$

$$x^2 = 2 x x_0 - x_0^2 - \frac{a^2}{b^2} y^2 + \frac{2a^2}{b^2} y y_0 + a^2\left(R^2 - \frac{y_0^2}{b^2}\right)$$

$$\underbrace{x_i^2}_{y} = \underbrace{\begin{bmatrix} 2x_i & -y_i^2 & y_i & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_0 \\ a^2/b^2 \\ \frac{2a^2}{b^2} y_0 \\ a^2\left(R^2 - \frac{y_0^2}{b^2}\right) - x_0^2 \end{bmatrix}}_{x}$$