$$\underline{\dot{x}} = A\underline{x} + \cancelto{0}{B\underline{u}}$$

Autonomous Linear Dynamical Systems

Gabriel Hugh Elkaim

Autonomous Linear Dynamical Systems

- Autonomous Linear Dynamics Systems

$\dot{\underline{x}} = A\underline{x} + \cancel{B\underline{u}}^{\,0}$

- Examples
- Higher order systems
- Linearization near equilibrium point
  - Linearization along a trajectory
- Linearization along trajectory

Autonomous Linear Dynamical Systems (1.4)

continuous time autonomous linear dynamical system

$$\dot{x} = Ax + \cancel{Bu}^{\,0,\text{ autonomous}}$$

C.T.I. $A \neq f(t)$.

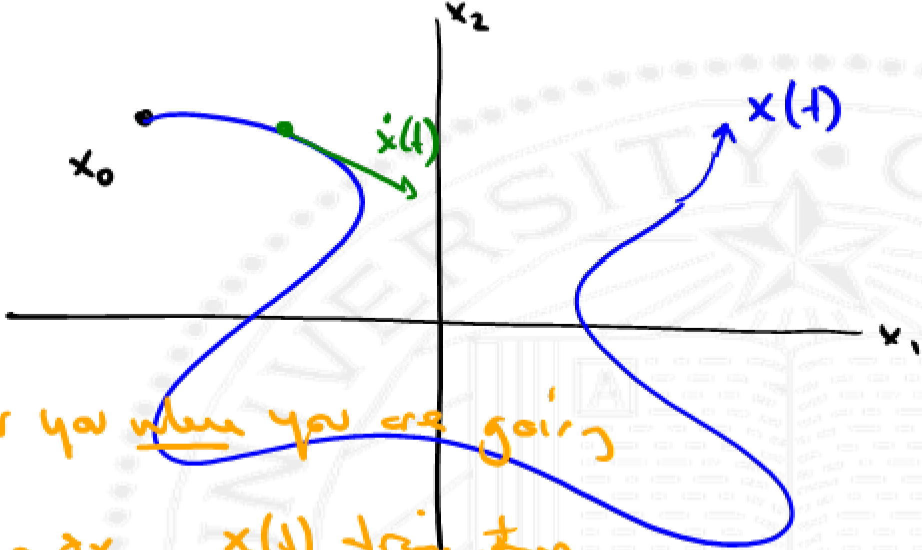
$$x : \mathbb{R} \to \mathbb{R}^n$$

$x(t) \in \mathbb{R}^n$ called "state" x_i called i^{th} state.

n – state dimension, order, number of states.

$A(t) \in \mathbb{R}^{n \times n}$ dynamics matrix.

Autonomous Linear Dynamical Systems (2.4)



A tells you where you are going

$\dot{x} = \frac{\partial x}{\partial t}$

$x(t)$ trajectory

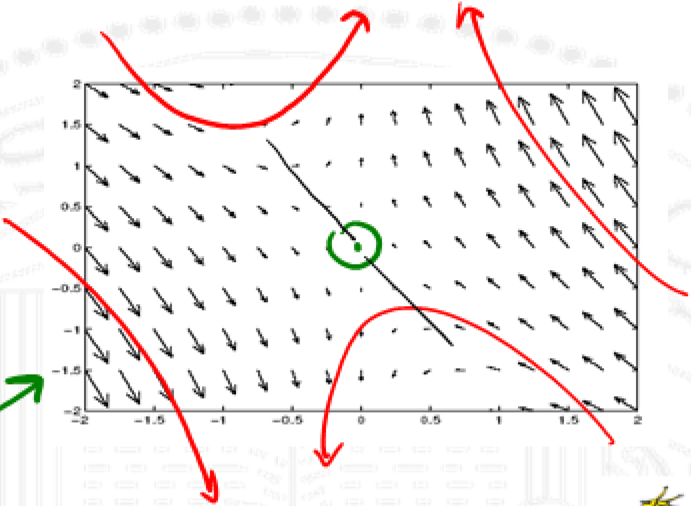
$\dot{x}(t)$ tangent to trajectory

Autonomous Linear Dynamical Systems (3.4)

$\dot{x} = Ax$

$\mathbb{R}^2$ $\mathbb{R}^2$

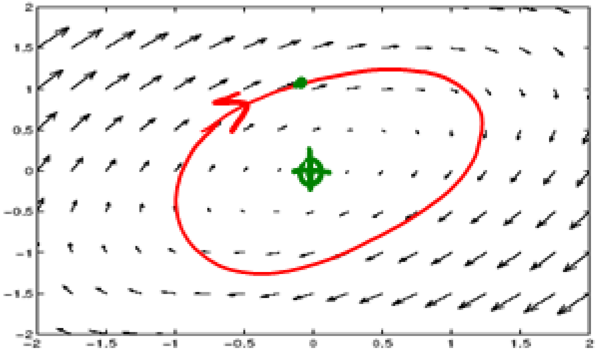
UNSTABLE



Autonomous Linear Dynamical Systems (4.4)

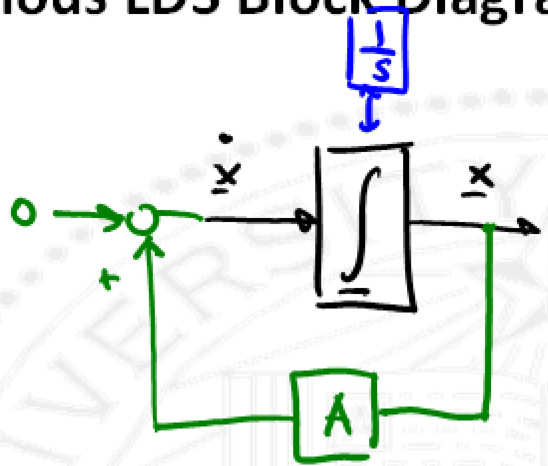
$\dot{x} = Ax$

PURELY OSCILLATORY

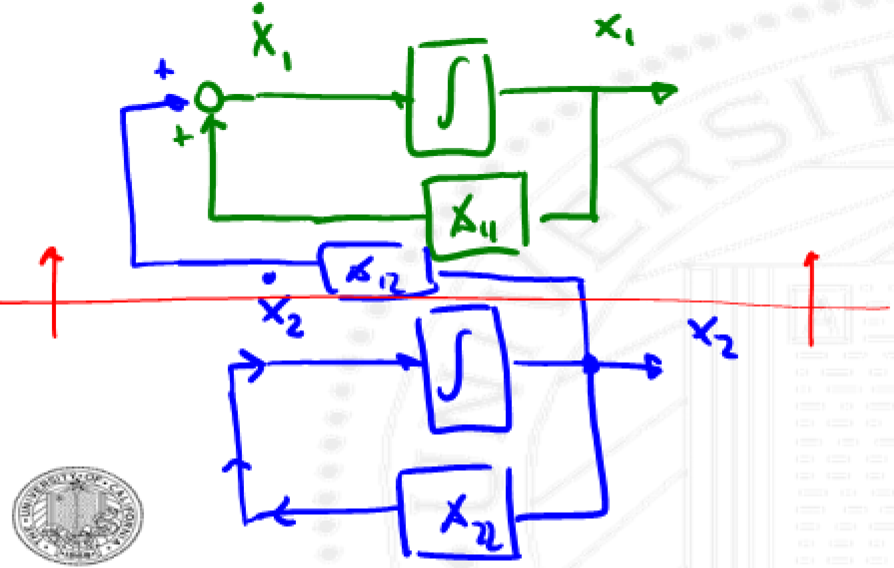


Autonomous LDS Block Diagram (1.3)

$\dot{x} = Ax$

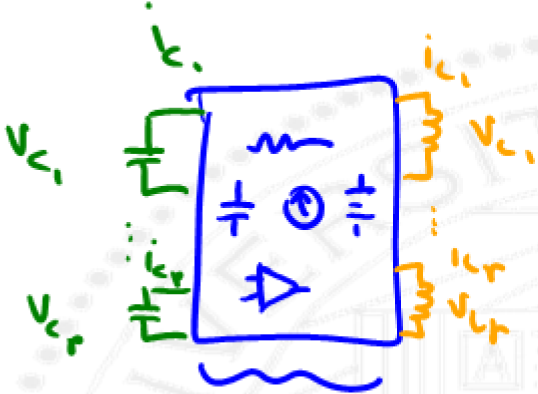


Autonomous LDS Block Diagram (2.3)



$$\dot{x} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] x$$

Autonomous LDS Block Diagram (3.3)

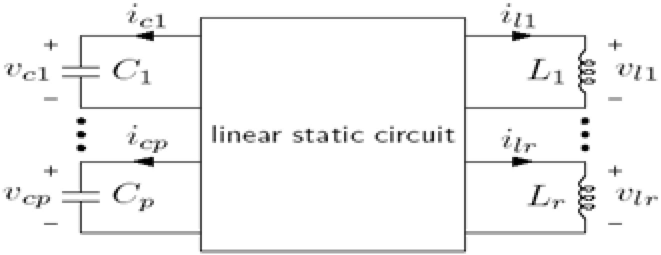


p - capacitors

r - inductors

F – linear static circuit

Linear Circuit Example (1.2)



$$\begin{bmatrix} i_c \\ v_l \end{bmatrix} = F \begin{bmatrix} v_c \\ i_l \end{bmatrix}$$

$$C \frac{dv_c}{dt} = i_c \qquad L \frac{di_l}{dt} = v_l$$

$$C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_p \end{bmatrix} \qquad L = \begin{bmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_r \end{bmatrix}$$

Linear Circuit Example (2.2)

$$x \triangleq \begin{bmatrix} v_c \\ i_L \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} C^{-1} i_c \\ L^{-1} v_L \end{bmatrix}$$

$$\dot{\underline{x}} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} F \underline{x}$$

$$\underline{\dot{x} = Ax}.$$

Chemical Reactions (1.3)

Chemical reaction involving n chemicals

x_i concentration of chemical i

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

A usually very sparse.

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

Chemical Reactions (2.3)

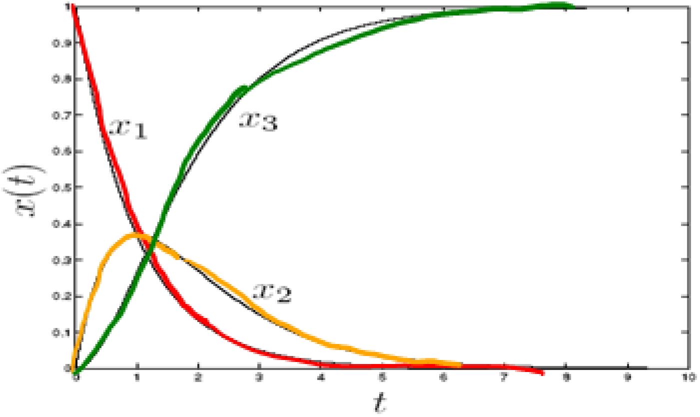
$k_1 = k_2 = 1$

$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad A \xrightarrow{1} B \xrightarrow{1} C$

$$\begin{array}{cc} \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} - \\ + \\ \end{array} \end{array}$$

$\boxed{0 \quad 0 \quad 0 \quad 0}$

$\dot{x}_2 = x_1 - x_2$



Chemical Reactions (3.3)

Finite-State Discrete-Time Markov Chain (1.3)

$z(t) \in \{1 \dots n\}$ random sequence with:

$$\text{Prob}\left(z(t+1) = i \mid z(t) = j\right) \triangleq P_{ij}$$

$P \in \mathbb{R}^{n \times n}$ matrix of transition probabilities

jump markov state.

$\text{Prob}\left(z(t) = 1,2,3\right) = [1 \; 1 \; 1 \; \circ\!\!-\!\!-\!\!\circ \,] \, p(t).$

$p(t+1) = P\,p(t) \rightarrow p(t+1) \neq p(t) P^T$

Finite-State Discrete-Time Markov Chain (2.3)

P is 'p-r-a.

nodes are the states

edges show transition probabilities

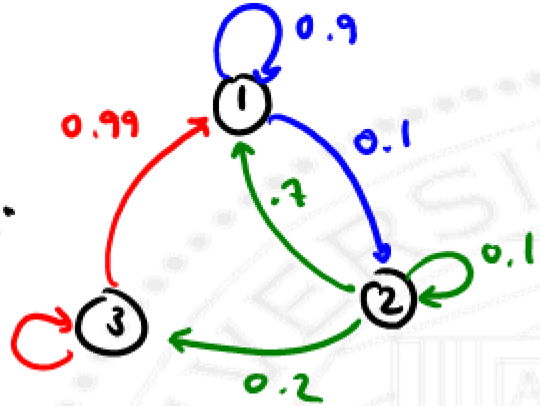
$\Sigma \, col(P) = 1$.

Finite-State Discrete-Time Markov Chain (3.3)

① "ok"

② "down"

③ "under repair"



$P^k \underline{p(0)}$ = prob.

Markov Chain Mixing

$$p(k+1) = \begin{bmatrix} 0.9 & 0.7 & 0.99 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0.01 \end{bmatrix} p(k)$$

Numerical Integration of Continuous System (1.3)

compute an approximate solution of $\dot{x} = Ax \quad x(0) = x_0$.

small time step, h.

over interval h, x doesn't change very much.

$$x(h+t) \simeq x(t) + h\dot{x}(t) = x(t) + hAx(t) = [I + Ah]x(t)$$

$$x(kh) \simeq [I + Ah]^k x(0) \leftarrow \text{FORWARDS EULER.}$$

$$x(t+h) \simeq [I - Ah]^{-1} x(t) \leftarrow \text{BACKWARDS EULER}$$

Numerical Integration of Continuous System (2.3)

Numerical Integration of Continuous System (3.3)

Higher Order LDS (1.3)

$x^{(k)} = A_{k-1} x^{(k-1)} + \dots + A_1 x^{(1)} + A_0 x \qquad x(t) \in \mathbb{R}^n$

$\dot{x} = Ax$ overloaded $\qquad x^{(n)}$ n^{th} derivative of x.

$x^{(0)} = x$

$x^{(1)} = \dot{x}$

$x^{(2)} = \ddot{x}$

$\vdots$

$$z \triangleq \begin{bmatrix} x \\ x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbb{R}^{nk}$$

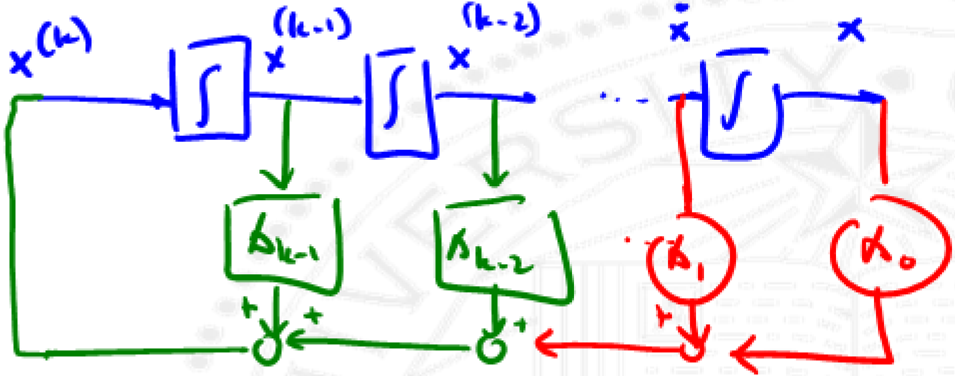
Higher Order LDS (2.3)

$$\dot{z} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ & 0 & I & 0 & \cdots & 0 \\ & & & I & \\ A_0 & A_1 & \cdots & & A_{k-1} \end{bmatrix}$$

← shift matrix

BLOCK COMPANION FORM

Higher Order LDS (3.3)



chain of integrators

Example: Mechanical Systems

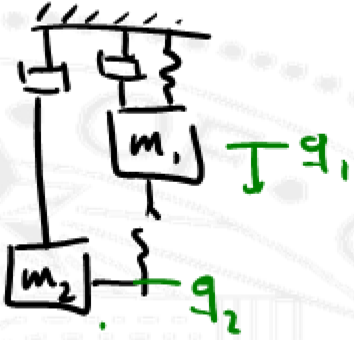
$$M\ddot{q} + D\dot{q} + Kq = 0$$

$q(t) \in \mathbb{R}^k$ generalized displacements

$M \in \mathbb{R}^{k \times k}$ diagonal $\begin{bmatrix} \ddots & & \\ & m_i & \\ & & \ddots \end{bmatrix}$

$K \in \mathbb{R}^{k \times k}$ stiffness matrix

$D \in \mathbb{R}^{k \times k}$ damping matrix



$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \quad \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}D \end{array}\right] \begin{bmatrix} q \\ \hline \dot{q} \end{bmatrix}$$

$$\dot{x} = Ax$$

Linearization near Equilibrium (1.3)

Non-linear time invariant differential equation

$\dot{x} = f(x) \qquad f: \mathbb{R}^n \to \mathbb{R}^n$

x_e some equilibrium point $\quad f(x_e) = 0. \quad x(t) = x_e \ \forall t.$

$x(t)$ near x_e : $\dot{x}(t) = f(x(t)) = f(x_e) + \underbrace{Df(x_e)}_{\text{Jacobian of } f}\underbrace{(x(t) - x_e)}_{\delta x}$

$\delta x \triangleq x(t) - x_e$

Linearization near Equilibrium (2.3)

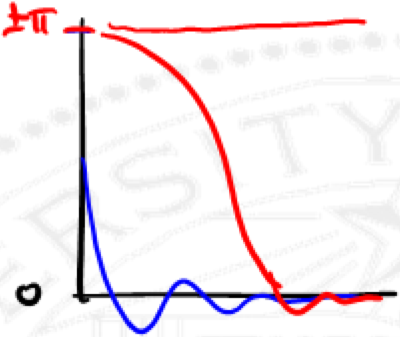
$$\dot{\underline{\delta x}} \simeq \underbrace{Df(x_e)}_{F} \underline{\delta x}$$

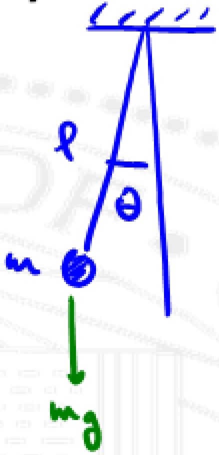
small signal model
linearized dynamics
perturbation equations

Linearization near Equilibrium (3.3)

$$m\ell^2\ddot{\theta} = -\ell m g \sin\theta$$

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ -\frac{g}{\ell}\sin\theta\end{bmatrix}$$





$$\sin x \simeq x$$

$$\delta\dot{x} = \begin{bmatrix}0 & 1\\ -\frac{g}{\ell} & 0\end{bmatrix}\delta x$$

MIDTERM covers everything up to ADS.

Open book
Open notes
Open videos (any year)
Open MATLAB / Calculators
Open wikipedia

- NO LIVING HELP
- NO SOLUTIONS.

24 HOURS

CANVAS

PRACTICE MIDTERM (HW #5)

AFTER CLASS ON FRIDAY 5/MAY 2

8 MONDAY 8/MAY 9

5 WEDNESDAY 10/MAY 8

Does Linearization "work"? (1.2)

$\dot{x} = -x^3$

$x_e = 0.$

$x(0) > 0$

$x(t) = \sqrt{\frac{1}{x_0^2} + 2t}$

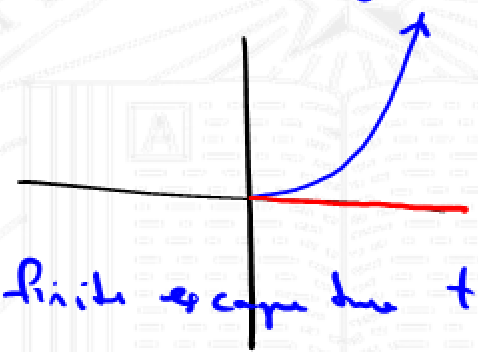
linearized system

$\dot{\delta x} = 0.$

$\dot{\delta z} = 0.$

$\dot{z} = z^3 \quad z(0) > 0$

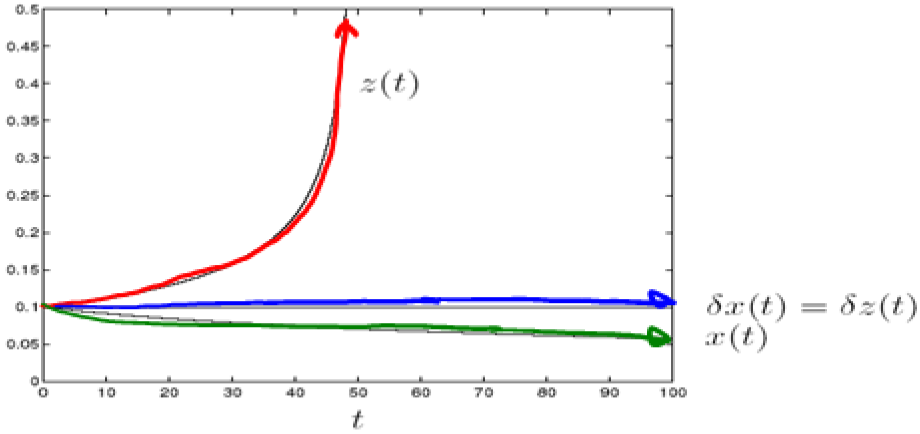
$z(t) = \sqrt{\frac{1}{z_0^2} - 2t}$



finite escape time $t = \frac{1}{2x_0^2}$

Does Linearization "work"? (2.2)

$x(0) = z(0) = 0.1$



Linearization along Trajectory (1.3)

$x_{traj} : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$

$\dot{x}_{traj}(t) = f(x_{traj}(t), t)$

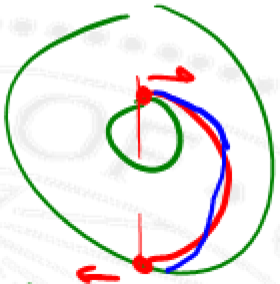
$x(t)$ is some other trajectory $\quad \dot{x}(t) = f(x(t), t)$

$x(t)$ is near $x_{traj}(t)$.

$$\frac{d}{dt}\underbrace{(x - x_{traj})}_{\delta x} = f(x,t) - f(x_{traj}, t) \approx \underbrace{D_x f(x_{traj}, t)(x - x_{traj})}_{\delta x}$$

Linearization along Trajectory (2.3)

$$\underline{\delta \dot{x}} \approx D_x f(x_{traj}, t)\,\underline{\delta x}$$



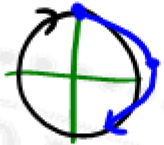
linearized or variational trajectory

orbital transfer

optimal control

Neighboring optimal control.

Linearization along Trajectory (3.3)



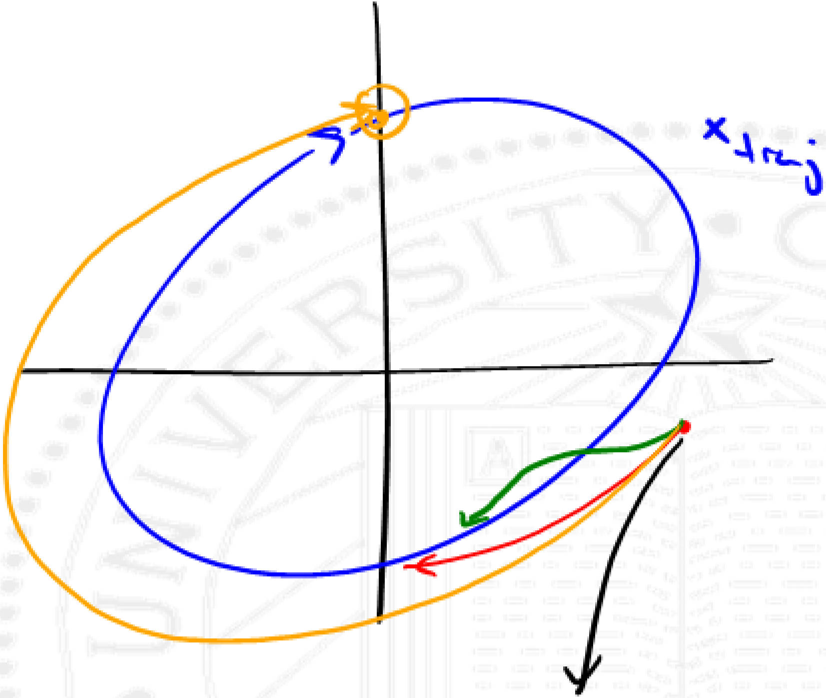
Linearized oscillator (EE)

x_{traj} T-periodic solution (NLDE)

$\dot{x}_{traj}(t) = f(x_{traj}, t)$ $\quad x_{traj}(t+T) = x_{traj}(t)$

$\dot{\delta x} = A(t)\,\delta x$ $\quad A(t) = Df(x_{traj}, t)$

T-periodic system "Floquet system"


x traj

Questions?