Linear Algebra Review

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Linear Algebra Review

- Vector Spaces, subspaces
- Independence, basis, dimension
- Range, nullspace, rank
- Change of coordinates
- Norm, angle, inner product.

Vector Spaces

- A *vector space* or *linear space* consists of:

a set $\mathcal{V}$

a vector sum $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ $\quad x+y \quad +(x,y)$

a scalar multiplication: $\times : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ $\quad \times(\alpha, v) \quad \alpha v$

a distinguished element $0 \in \mathcal{V}$

Vector Space Properties

$$x + y = y + x \quad \forall\, x, y \in \mathcal{V} \quad (+\ \text{commutative})$$

$$(x+y)+z = x+(y+z) \quad \forall\, x, y, z \in \mathcal{V} \quad (+\ \text{associative})$$

$$0 + x = x \quad \forall\, x \in \mathcal{V} \quad (\text{additive identity})$$

$$\forall\, x \in \mathcal{V} \quad \exists (-x) \in \mathcal{V} \text{ such that } x + (-x) = 0$$

(existence of an additive inverse)

$$(\alpha\beta)x = \alpha(\beta x) \quad \forall\, \alpha, \beta \in \mathbb{R},\ x \in \mathcal{V}$$

(scalar multiplication associative)

$$\alpha(x+y) = \alpha x + \alpha y \quad \text{(right distributive rule)}$$

$$(\alpha+\beta)x = \alpha x + \beta x \quad \text{(left distributive rule)}$$

$$1x = x, \quad \forall\, x \in \mathcal{V}$$

Vector Space Examples

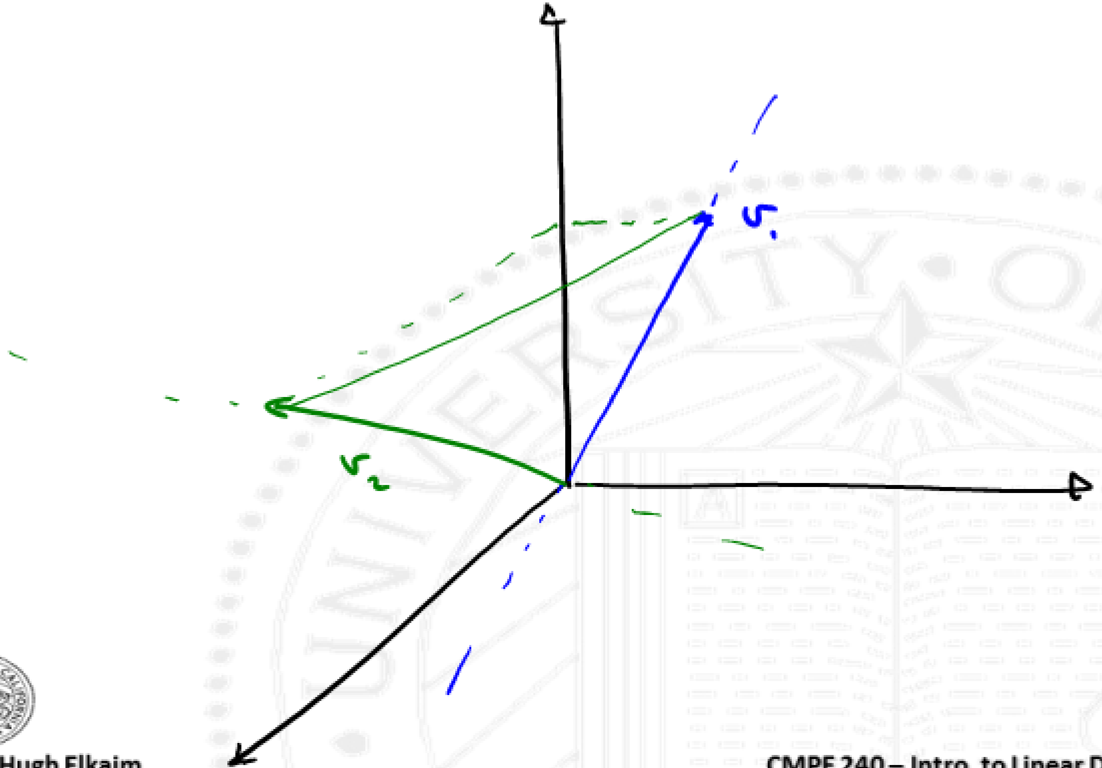
$V_1 = \mathbb{R}^n$ w/ standard elementwise vector addition and scalar multiplication

$V_2 = \{0\}$ where $\{0\} \in \mathbb{R}^n$ ← trivial

$V_3 = \text{span}\{v_1, v_2, \ldots v_k\}$ where

$\text{span}\{v_1, v_2 \ldots v_k\} \triangleq \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$

$v_1 \ldots v_k \in \mathbb{R}^n$



v_1
v_2

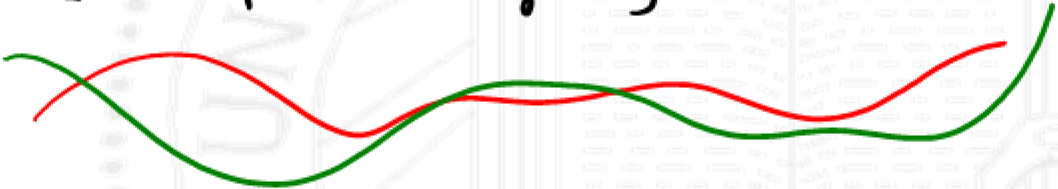
Vector Spaces of Functions

$V_4 = \{x : \mathbb{R}_+ \longrightarrow \mathbb{R}^n \mid x \text{ is differentiable}\}$ where vector sum is a sum of functions

$(x+z)(t) = x(t) + z(t)$

scalar multiplication $(\alpha x)(t) = \alpha x(t)$

a <u>point</u> in V_4 is a trajectory in $\mathbb{R}^n$



Subspaces

A subspace of a vector space is a subset of a vector space which itself is also a vector space.

Subspace is closed under vector addition and scalar multiplication

V_1, V_2, V_3 are subspaces of $\mathbb{R}^n$

$V_5 = \{ x \in V_4 \mid \dot{x} = Ax \}$ points in V_5 are trajectories of $\dot{x} = Ax$.

Independent Set of Vectors

Property of a <u>set</u> of vectors $\{v_1 \dots v_k\}$ <u>is</u> independent

if: $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0 \rightarrow \alpha_1 \dots \alpha_k = 0$.

No vector v_i can be expressed as a linear combination of the other vectors.

ex: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \longrightarrow$

$$\alpha_1 v_1 + \dots + \alpha_k v_k = \beta_1 v_1 + \dots + \beta_k v_k \longrightarrow \alpha_1 = \beta_1 \dots \alpha_k = \beta_k$$

Basis and Dimension

$\{v_1 \cdots v_k\}$ is a <u>basis</u> for a vector space V iff:

$\{v_1 \cdots v_k\}$ span V $\quad V = \text{span}\{v_1 \ldots v_k\}$

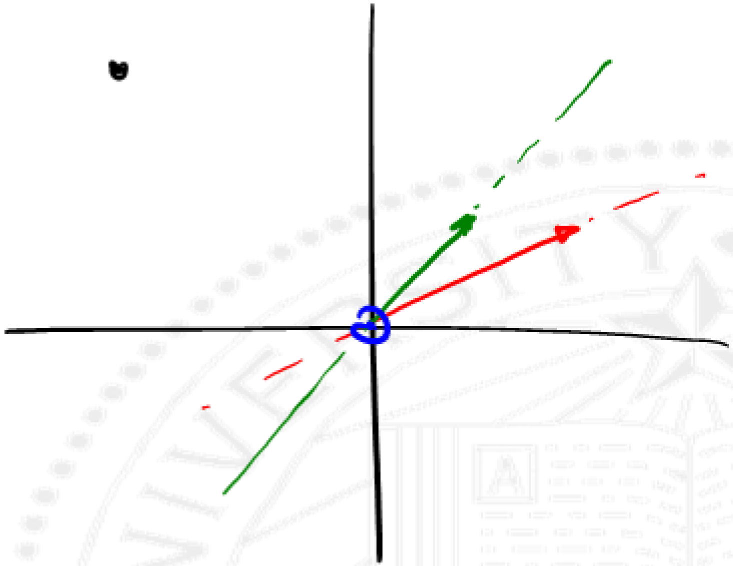
<u>and</u> $\{v_1 \cdots v_k\}$ is independent

every $v \in V$ can be uniquely expressed

$$v = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

$\text{Dim}(V) \triangleq$ # of v_k's that form a basis

"Cardinality"



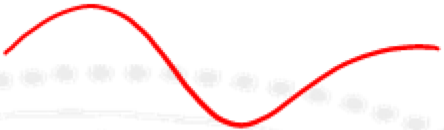
Jacobian

GRADIENT

$\downarrow$

$$\underline{x} \longrightarrow \frac{\partial x}{\partial t}$$

$\underline{x}(1)$



$$\frac{\partial \underline{x}}{\partial t} \longrightarrow D\underline{x}$$

$\uparrow$

"Jacobian"

Nullspace of a Matrix

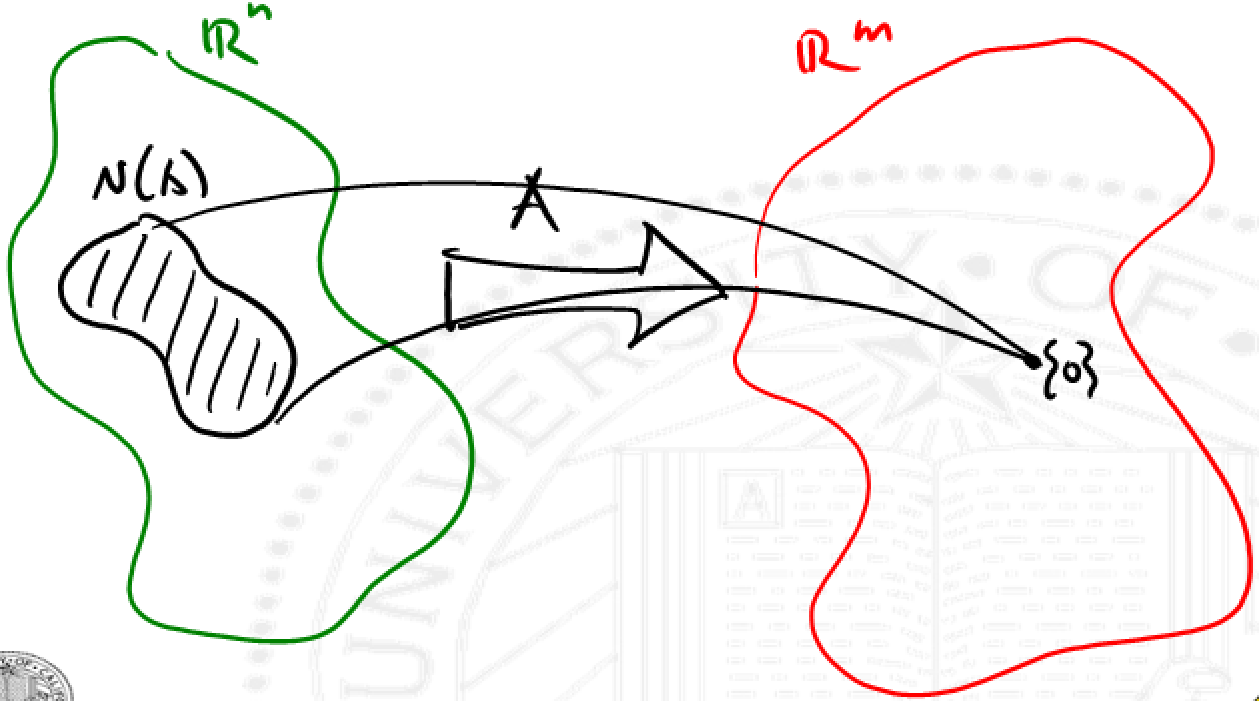
$A \in \mathbb{R}^{m \times n}$

null space
"kernel"

$$N(A) \triangleq \{x \in \mathbb{R}^n \mid Ax = \underset{\mathbb{R}^m}{0}\}$$

$N(A)$ is a set of vectors mapped to zero by $y = Ax$

$N(A)$ is a set of vectors that are orthogonal to all rows of A.


$\mathbb{R}^n$
$\mathbb{R}^m$
$N(A)$
A
$\{0\}$

Zero Nullspace

$\mathbb{R}^n$

$\downarrow$

$N(A) = \{0\}$ — "one to one"

$y = Ax$

$x = By = BAx = Ix = x$

$\underline{x}$ can $\underline{\text{always}}$ be determined from $y = Ax$

$y = Ax$ transformation loses $\underline{\text{no}}$ information

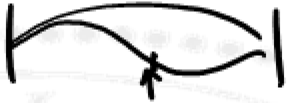
map from $x \to y$ different x's map to different y's.

$\det(A^T A) \neq 0$. { perfect decoder

A has a LEFT inverse $B \in \mathbb{R}^{n \times m}$ such that $BA = I$.

Interpretations of Nullspace

$y = Ax$ $\quad z \in N(A)$



$y = Ax$ is a measurement $\longrightarrow$ z is undetectable by my sensors

x and $x+z$ are indistinguishable

$N(A)$ is the ambiguity in x from measurement $y = Ax$.

$y = Ax$ is the output from input x.

z is input w/ no result

$x+z$ and x have the same result.

$N(A)$ freedom of choice

Range of a Matrix

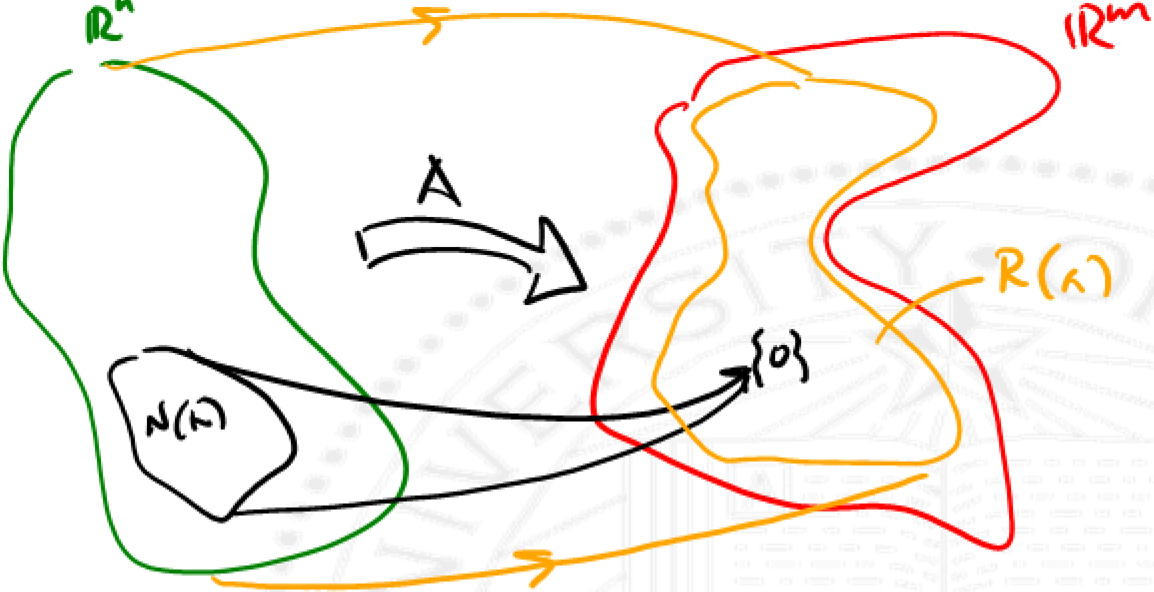
$R(A)$: range of A

$$R(A) \triangleq (Ax \mid x \in \mathbb{R}^n) \in \mathbb{R}^m$$

Set of all values that can be "hit" by $y = Ax$

Span $\{ a_1, a_2, \ldots, a_n \}$

Set of vectors $\{y\}$ for which $y = Ax$ can be solved.



$\mathbb{R}^n$
$\mathbb{R}^m$
A
$R(A)$
$N(A)$
$\{0\}$

"*Onto*" Matrices

$y = Ax \quad x = B y_{m}$

$y = AB y_m = I y_m$

$y = y_m$

A is called "onto" if $R(A) = \mathbb{R}^m$.

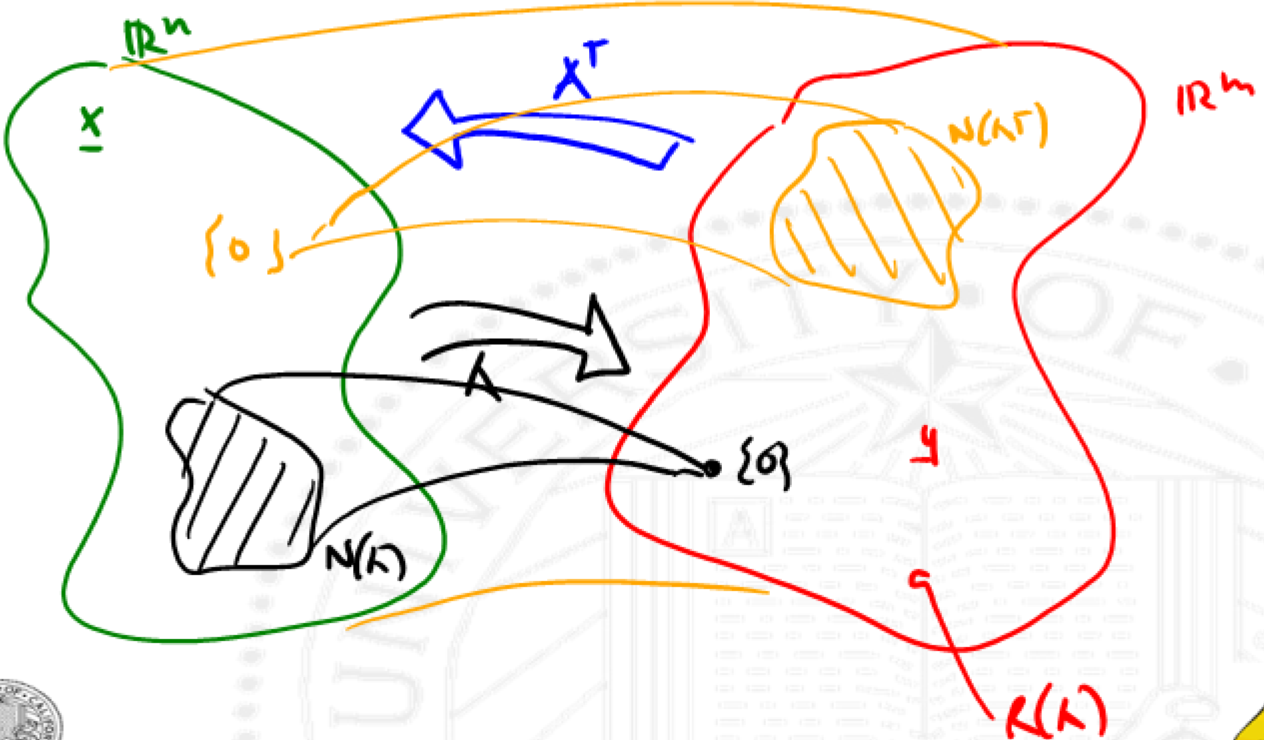
$y = Ax$ can be solved for any $\{y\}$.

Columns $\{a_1, a_2, \ldots, a_n\}$ span $\mathbb{R}^m$

A has a RIGHT inverse $B \in \mathbb{R}^{n \times m}$ $AB = I$

rows of A are independent $\{\tilde{a}_i^T\}$

$N(A^T) = \{0\}$ and $(AA^T) \neq 0$.



$\mathbb{R}^n$
$\underline{x}$
A^T
$\{0\}$
$N(A^T)$
$\mathbb{R}^n$
A
$\{0\}$
$N(A)$
$\underline{y}$
$R(A)$

Interpretations of Range

$$Ax = \sum_{i=1}^{n} x_i a_i = \text{span}\{a_1, \dots, a_n\} \qquad A \in \mathbb{R}^{m \times n}$$

$$N(A) := \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$R(A) = \{\underbrace{Ax}_{\text{image of } A} \mid x \in \mathbb{R}^n\}$$

$v \in R(A) \qquad w \notin R(A)$

$y = Ax \leftarrow$ measurement

$y = v$ possible or consistent sensor measurement.

$y = w$ impossible sensor measurement / inconsistent signal

$y = Ax$ is an output/result

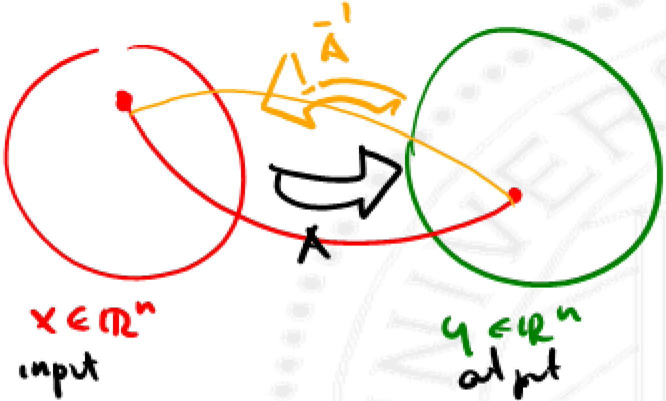
$y = v$ is a possible result or output

$y = w$ is an impossible result or output

Square matrices $(n=m)$ Inverse

$\det(A) = \det(A^T)$
$\det(AB) = \det(A)\det(B)$

$A \in \mathbb{R}^{n\times n}$ invertible (non-singular) if $\det(A) \neq 0$.



$N(A) = \{0\}$

"onto"

$R(A) = \mathbb{R}^n$

$\det(AA^T) = \det(A^TA) \neq 0$.

A has both a left and right inverse such that $\underline{A^{-1}A = AA^{-1} = I}$.

Columns of $A \to \{a_1 \ldots a_n\}$ are a basis for $\mathbb{R}^n$

$y = Ax$ has a UNIQUE solution for every $y \in \mathbb{R}^n$

Interpretations of Inverse

$A \in \mathbb{R}^{n \times n}$ has an inverse $B = A^{-1} \in \mathbb{R}^{n \times n}$

- mapping associated with B undoes the mapping associated w/ A. (pre- or post)

$x = By$ as a perfect (pre- or post) equalizer for the channel $y = Ax$

$x = By$ is the <u>unique</u> solution to $y = Ax$

Dual Basis Interpretation

Let a_i be a column of A Let $\tilde{b}_i^T$ row of $B = A^{-1}$

$y = Ax = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ $x_i = \tilde{b}_i^T y$.

$y = \sum_{i=1}^{n} (\tilde{b}_i^T y) a_i$ ← extract x_i's.

$\{\tilde{b}_1 \dots \tilde{b}_n\}$ and $\{a_1 \dots a_n\}$ dual basis for A.

rows of B columns of A

$y = \sum_{i=1}^{n} x_i \underbrace{\tilde{b}_j^T a_i}$ → $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

Rank of a Matrix

$A \in \mathbb{R}^{m \times n}$ $\quad \text{rank}(A) \triangleq \dim(\mathcal{R}(A))$

all possible combination of columns of A

(non-trivial) facts

$\text{rank}(A) = \text{rank}(A^T)$ ← not easy to prove

$\text{rank}(A)$ is the maximum # of independent columns or rows of A.

$\text{rank}(A) \leq \min(m, n)$

RANK NULITY THEOREM

$$\boxed{\text{rank}(A) + \dim(\mathcal{N}(A)) = n}$$

Conservation of Dimension

$\underline{\mathrm{rank}(A) + \dim(N(A)) = n}$:

→ rank(A) dimension of set "hit" by $y = Ax$

→ dim (N(A)) dimension of set x that are crushed to {0} by y=Ax

conservation of dimension: each dimension of input (x) is either crushed to {0} or shows up in the output (y).

n is the # of degrees of freedom in x

dim (N(A)) is # of degrees of freedom lost in $y = Ax$.

rank(A) # of degrees of freedom in y.

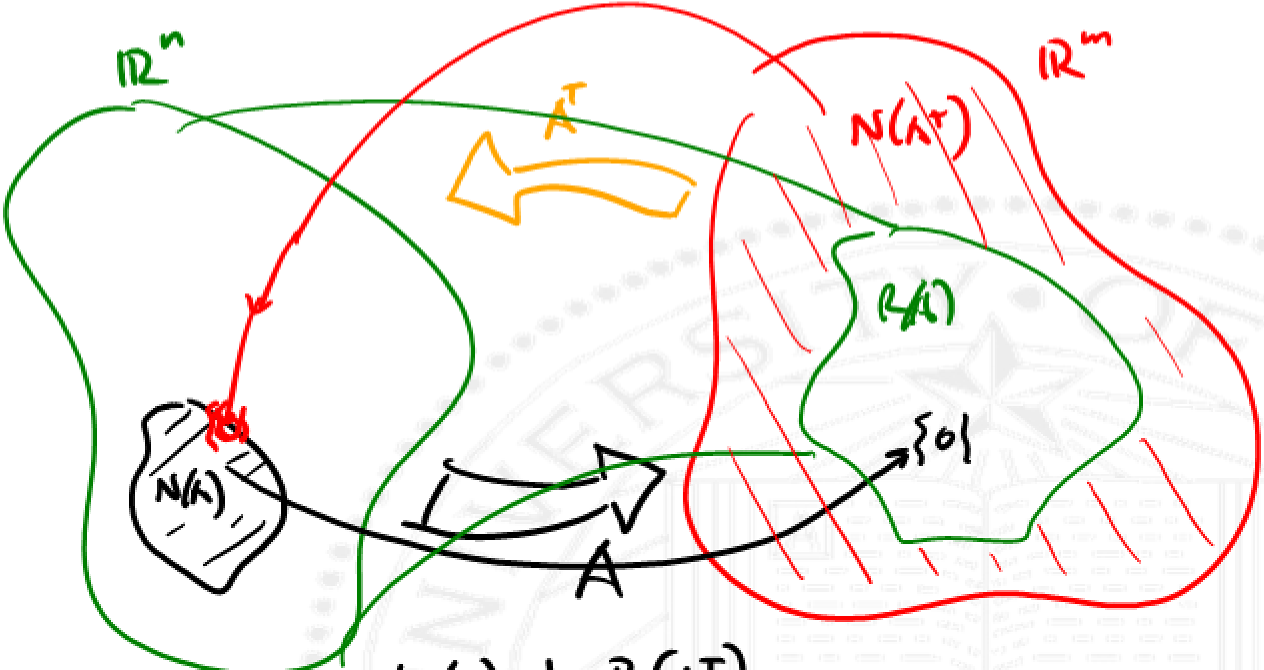
'Coding' Interpretation of Rank

$\mathrm{rank}(BC) \leq \min(\mathrm{rank}(B), \mathrm{rank}(C))$

if $A = BC$ $\quad B \in \mathbb{R}^{m\times r}$ & $C \in \mathbb{R}^{r\times n}$

$$x \xrightarrow{n} \underset{10^6\times 10^6}{\boxed{A}} \xrightarrow{m} y \quad \triangleq \quad x \xrightarrow{n} \underset{10^6\times 1000}{\boxed{C}} \xrightarrow{r} \underset{1000\times 10^6}{\boxed{B}} \xrightarrow{m} y$$

$\mathrm{rank}(A) \leq r.$



$$N(A) \perp R(A^T)$$

$$\dim(R(A)) = \dim(R(A^T)) = \operatorname{rank}(A) = \operatorname{rank}(A^T)$$

Application: fast matrix-vector multiplication

Full Rank Matrices

Change of Coordinates (1.3)

Change of Coordinates (2.3)

Change of Coordinates (3.3)

(Euclidean) norm

RMS value and (Euclidean) distance

Inner Product

Parallelogram Equality

Cauchy-Schwartz Inequality

Special cases of Cauchy-Schwartz

Half-space Interpretation

Questions?