

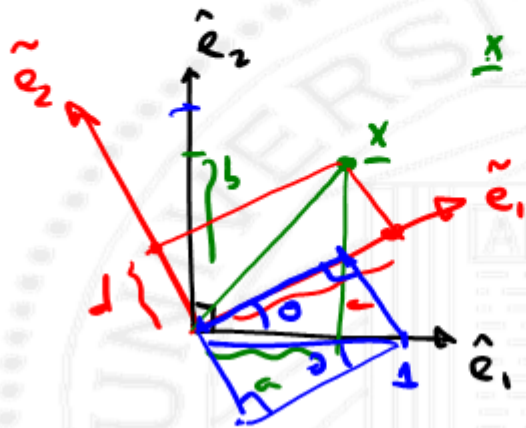
Linear Functions and Examples

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Questions?

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m \iff f(x) = Ax \quad \begin{matrix} \mathbb{R}^{m \times n} \\ \in \mathbb{R}^n \end{matrix}$$



$$\underline{x} = a\hat{e}_1 + b\hat{e}_2$$

$$\underline{y} = c\tilde{e}_1 + d\tilde{e}_2$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix}$$

\tilde{e}_1



Graviometer Prospecting

voxels



$$l$$

$$g = \Delta x$$

$$x_j = \rho_j - \rho_{avg}$$

g_i = measured
gravity
gradient

$g_i - g_{avg}$

$$x_j < 0$$

salt water

$$x_j > 0$$

reservoir / mineral deposit

$$x_j \ll 0$$

$g \rightarrow s$

10^{-7} g's

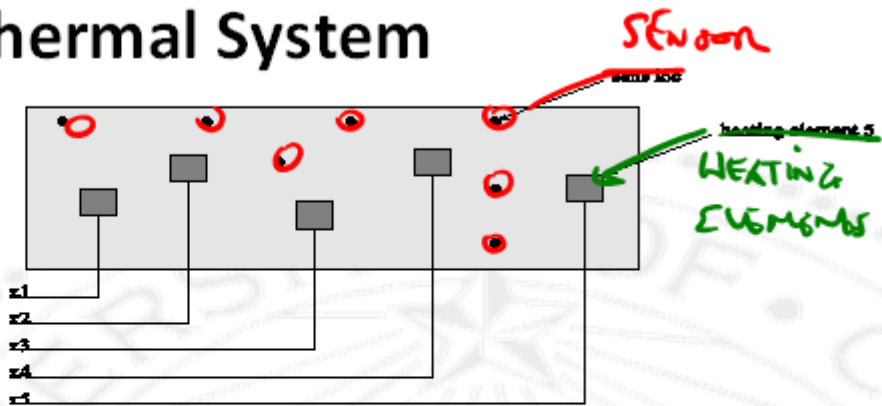


x_j - pair j 'th heating element

Thermal System

y_i - change in steady state temp @ location i

$$y = bx$$



Diffusion Process / Poisson ζq^2



Illumination with Multiple Lamps

LAMPS

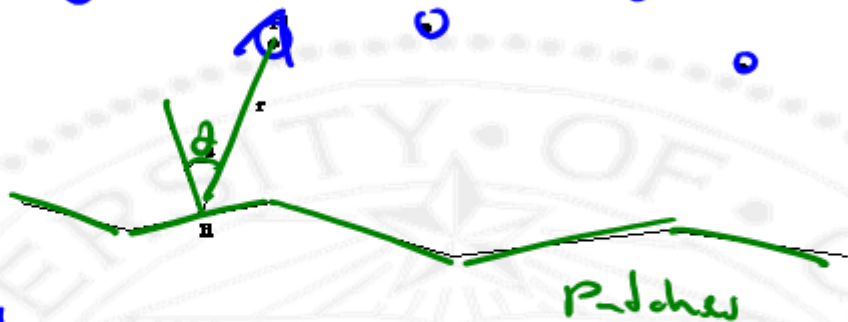
n -lamps

m -small patches no shadows

x_j - power to j th lamp

y_i - illumination on patch i

$$a_{ij} = r_{ij}^{-2} \max(\cos \theta_{ij}, 0)$$



Signal and Interference Power in Wireless System

w - transmitter/receiver pairs

P_j - power of the j^{th} transmitter

S_i - received signal power at i^{th} receiver



G_{ij} path gain from $j \rightarrow i$

$$S_i / P_j \triangleq \text{SNR}$$

$$S = A p$$

G_{ij} if $i=j$
 \emptyset otherwise

$$z = B p$$

G_{ij} if $i=j$
 G_{ij} otherwise



Cost of Production (1.2)

x_j price per unit production input j

y_i cost per unit of product i

a_{ij} unit of production input required to manufacture one unit of product i

q_i quantity of the product

r_j total quantity of production input j that is needed

$$y = Ax$$



Cost of Production (2.2)

$$r = A^T q$$

$$y = Ax$$

$$r^T x = (A^T q)^T x = q^T Ax = q^T y$$



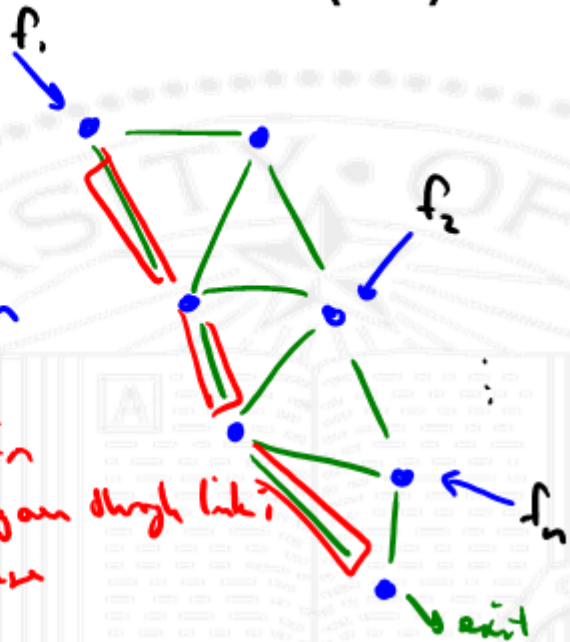
Network Traffic and Flows (1.2)

n - flows w/ nodes $f_1 \dots f_n$

t_i - traffic on link i
 Σ flows that pass through
link i

flow link incidence matrix

$$\Delta_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes through link } i \\ 0 & \text{otherwise} \end{cases}$$



Network Traffic and Flows (2.2)

$d_1 \dots d_n$ link delays

$l_1 \dots l_n$ latency (total time travel) - flows

$$t = A f$$



$$l = A^T d$$

delays in the network

$$f^T l = f^T A^T d = (A f)^T d = t^T d$$

A^T has a very interesting physical interpretation



Linearization

$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ differentiable @ $x_0 \in \mathbb{R}^n$ then:

if \underline{x} near $x_0 \rightarrow f(x)$ very near $f(x_0) + Df(x_0)(x-x_0)$

$$Df(x_0)_{ij} \triangleq \left. \frac{\partial f}{\partial x_j} \right|_{x=x_0} \leftarrow \text{Jacobian}$$

$y = f(x)$ ← n-to-n-linear

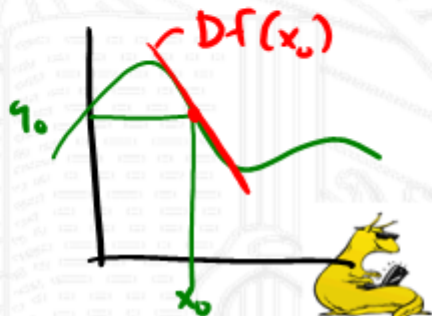
$$y_0 = f(x_0)$$

$$\delta x \triangleq x - x_0$$

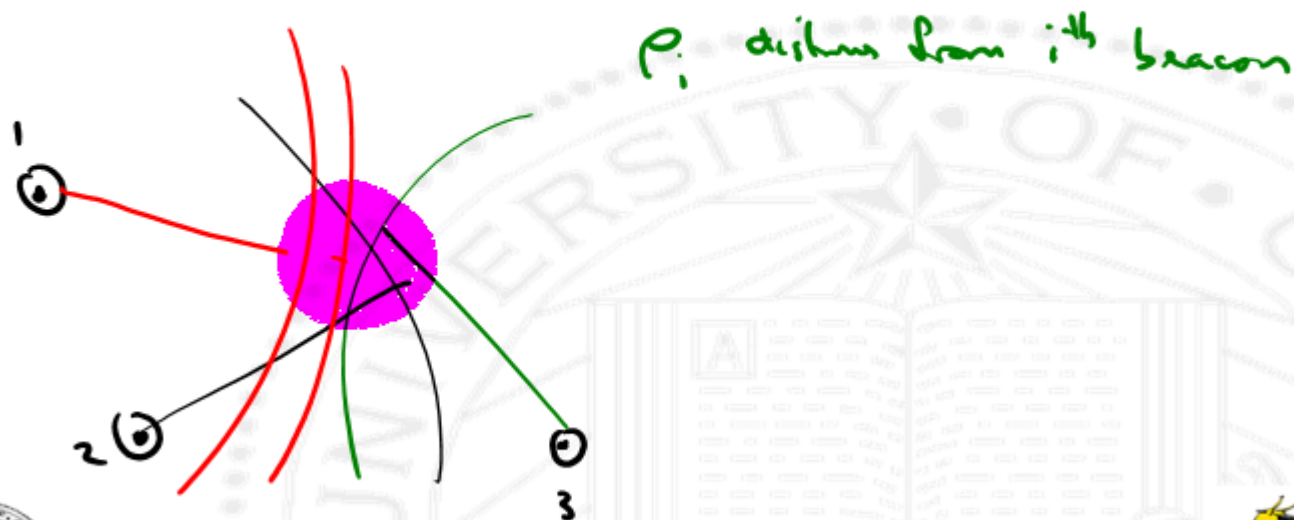
$$\delta y \triangleq y - y_0$$

$$\delta y \approx Df(x_0) \delta x$$

↑
LINEAR



Navigation by Ranging (1.2)



Navigation by Ranging (2.2)

$\rho \in \mathbb{R}^q$ is a non-linear function of $(x, y) \in \mathbb{R}^2$

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2} \quad (p, q)$$

$$\delta p \approx h \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

$$a_{i1} = \frac{x_0 - p_i}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

$$a_{i2} = \frac{y_0 - q_i}{\text{same}}$$



Broad Categories of Applications

- Linear model or function $y=Ax$
- Some broad categories of applications:
 - Estimation or inversion
 - Control or design
 - Mapping or transformation

(this list is not exclusive; can have combinations)



Estimation or Inversion

y_i is the i^{th} measured or sensor reading (which we know)

x_j is a parameter to be estimated/determined

G_{ij} sensitivity of j^{th} parameter to i^{th} sensor

→ find x , given y .

→ find all x 's that are consistent w/ y .

→ if no such x exists, note that $y = Gx$,

find the best x that is "most consistent" with y .

$y = Gx + d \leftarrow \text{noise}$



Control or Design

x is a number of design parameters or control inputs we can choose.

y is the output or result

find x such that $y \rightarrow y_{des}$

find all x 's that produce $y \rightarrow y_{des}$

Among all such x 's that give $y \rightarrow y_{des}$

choose the best one.



Mapping or Transformation

x is mapped or transformed by A into y

determine if there is an $x \rightarrow y$

Find an x that maps into y

Find all x 's that map into y .

decode or undo transformation.



Matrix Multiplication as Mixture of Columns

$$y = Ax \quad A \in \mathbb{R}^{m \times n}$$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}$$

$$a_j \in \mathbb{R}^m$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

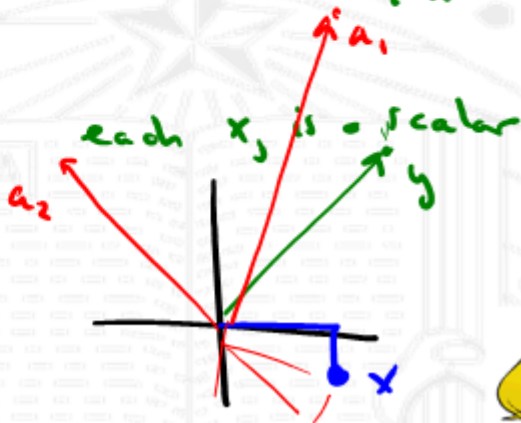
$$y = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

each x_j is a scalar

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$y = Ax = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$



Unit Vectors

$x = e_j$ the j^{th} unit vector

$$e_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$e_2 \triangleq \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

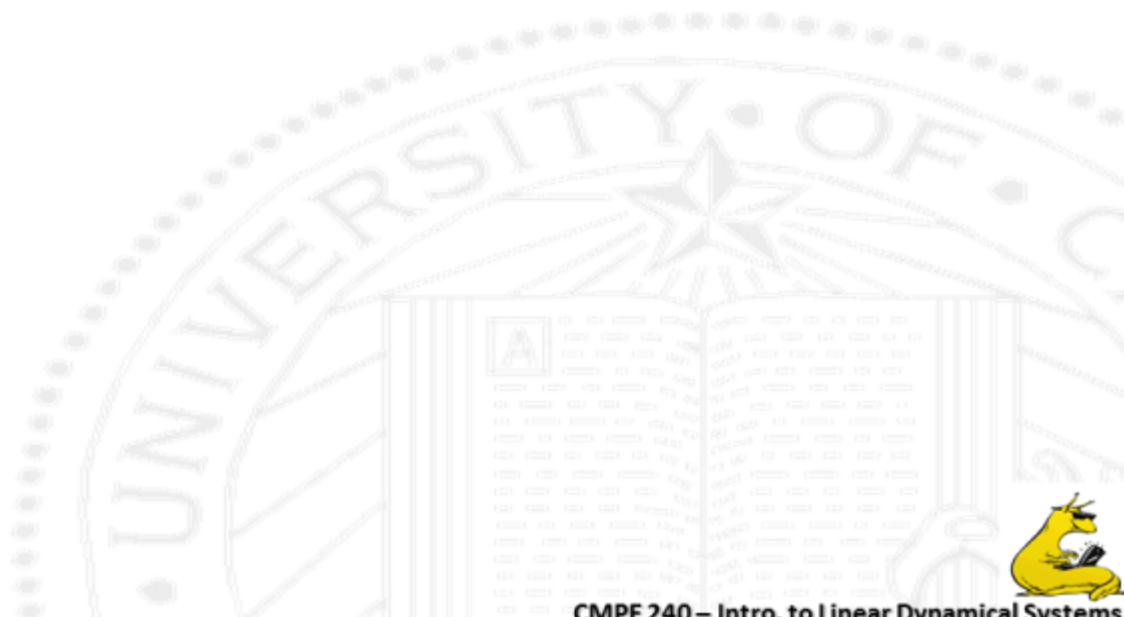
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$$e_n \triangleq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

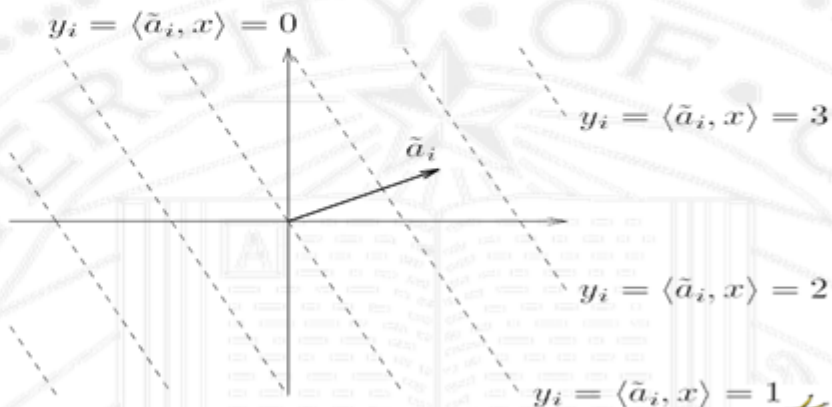
$Ae_j = a_j$ ← j^{th} column of A



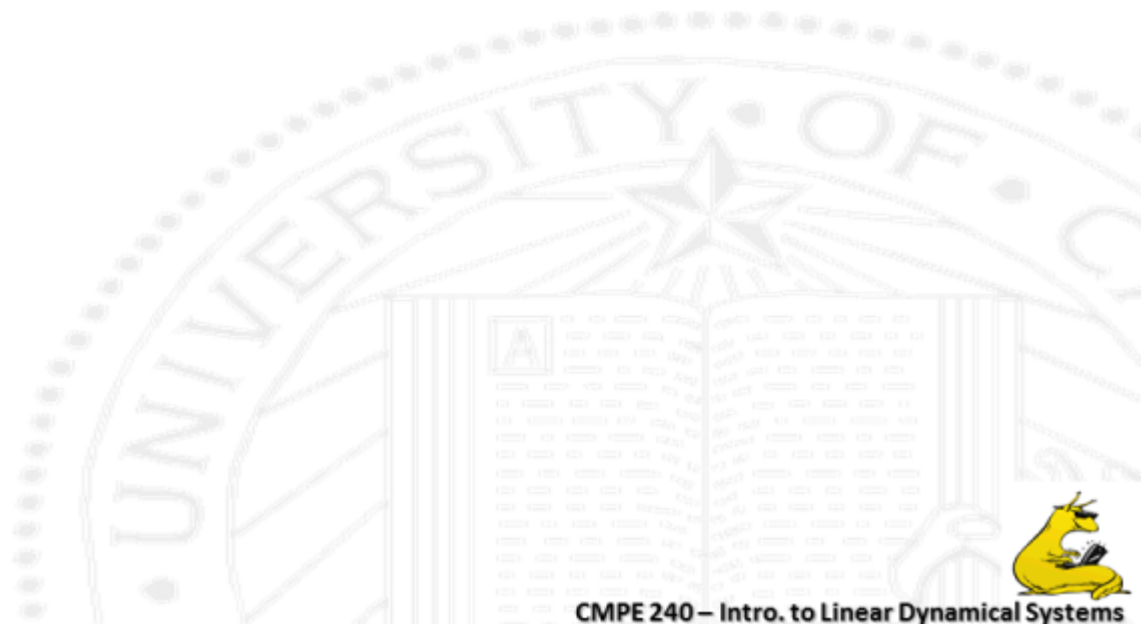
Matrix multiplication as inner product with rows



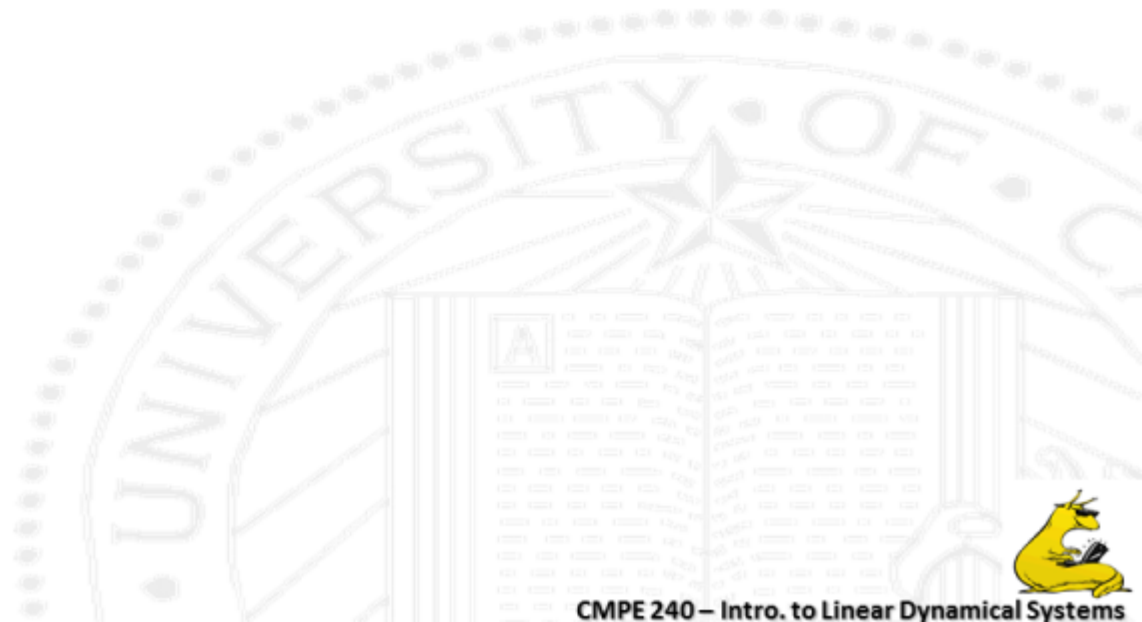
Geometric Interpretation



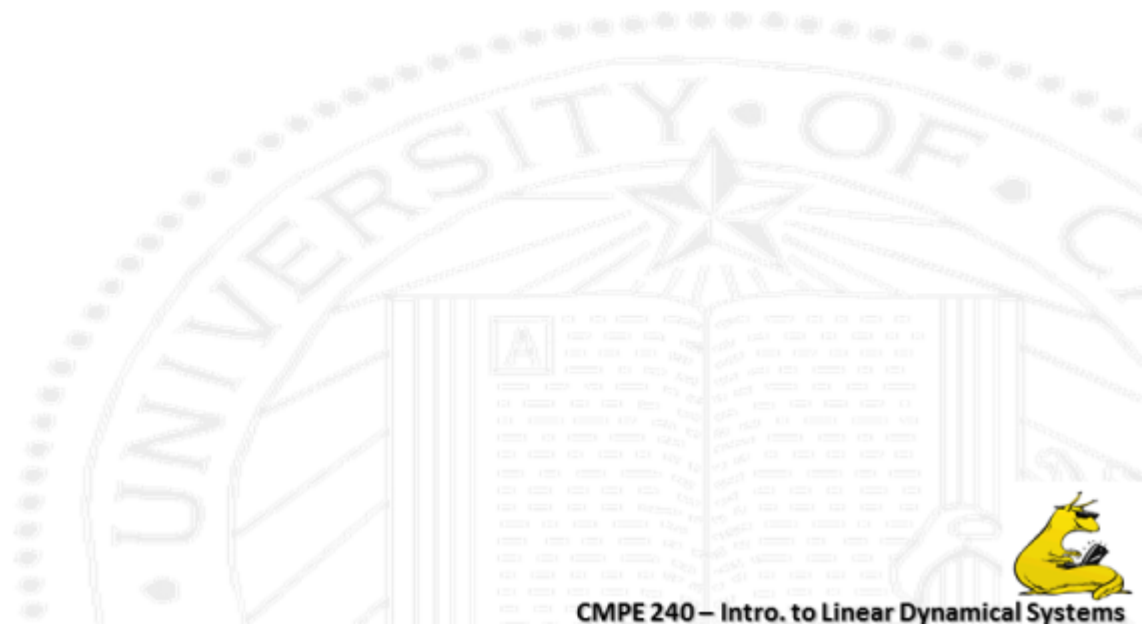
Block Diagram Representation



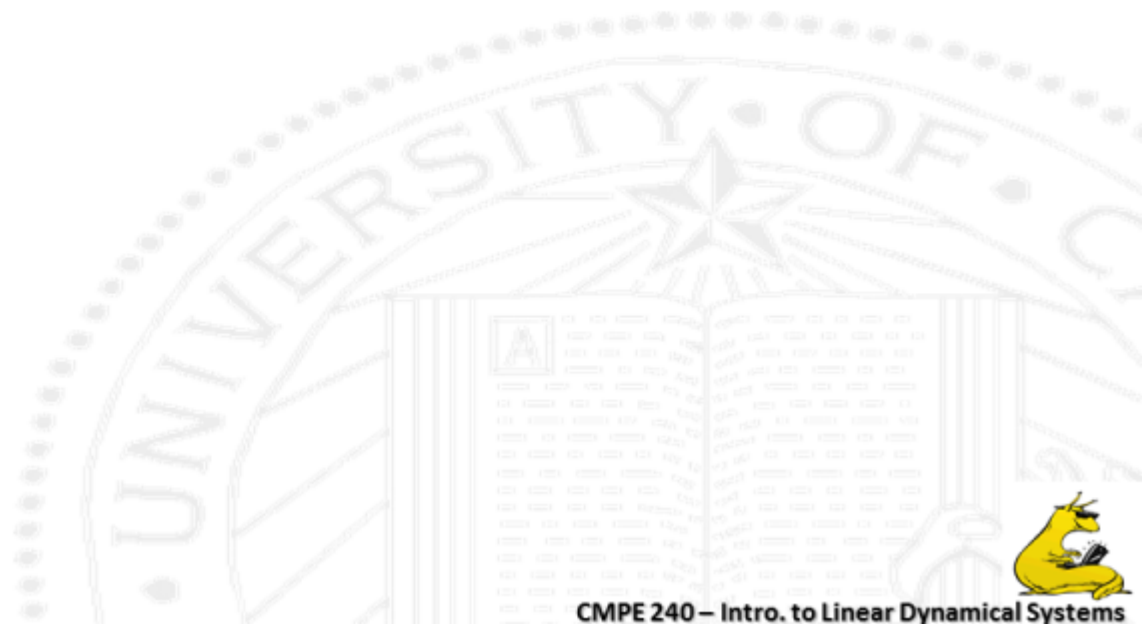
Example: Block Upper Triangular



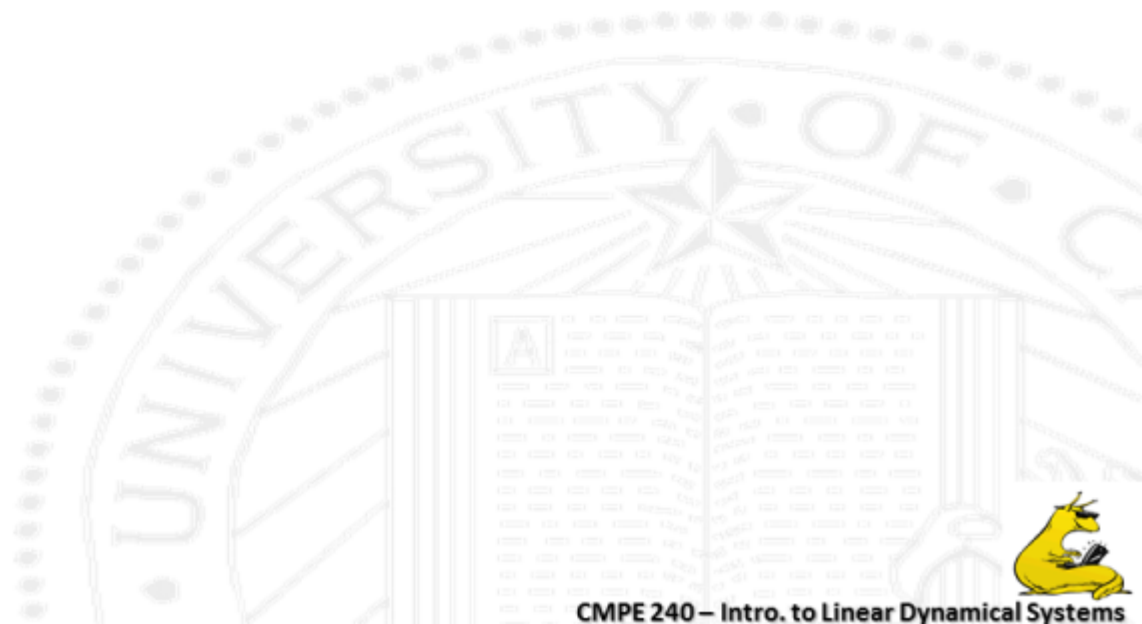
Matrix Multiplication as Composition



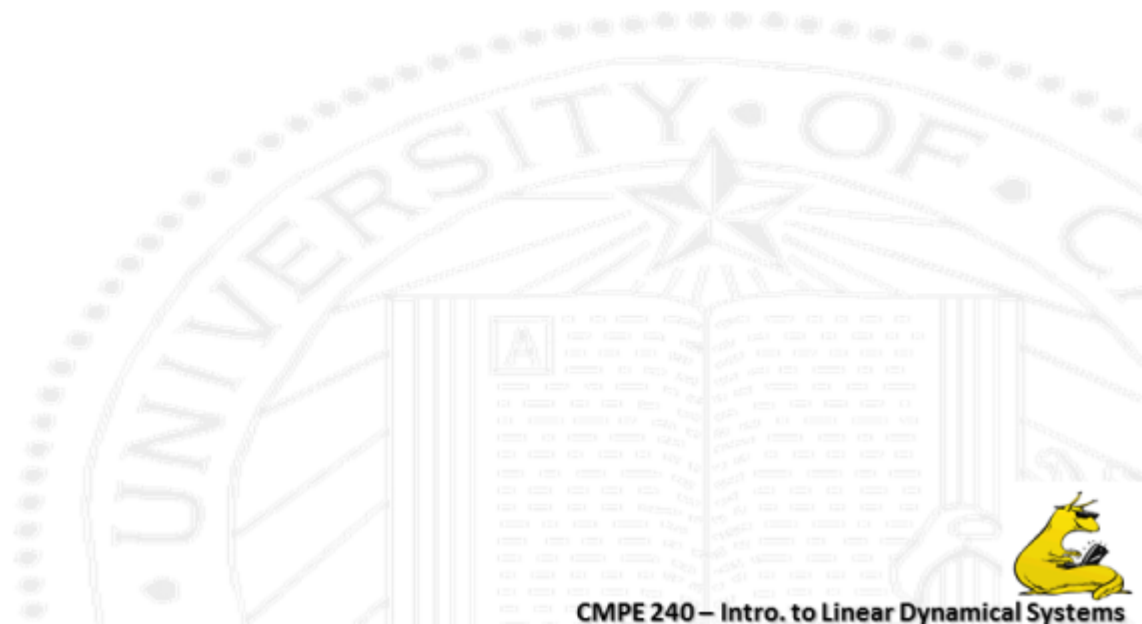
Column and Row Interpretations



Inner Product Interpretation



Matrix Multiplication Interpretation via Paths



Questions?





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CMPE 240 – Intro. to Linear Dynamical Systems