# Linear Functions and Examples

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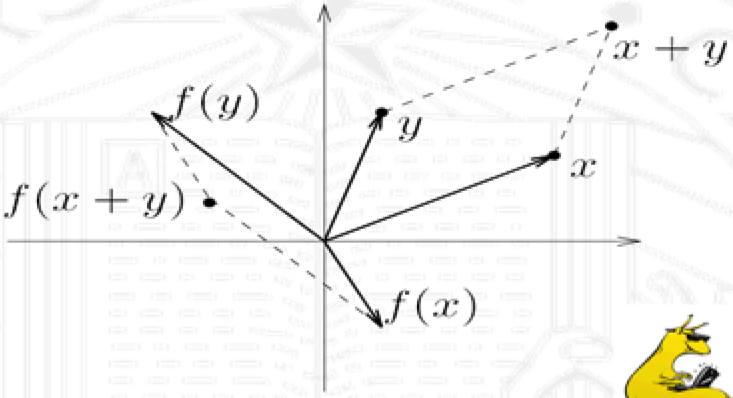
# Linear Functions and Examples

- Linear equations and functions
- Engineering Examples
- Interpretations

# Linear Equations

# Linear Functions



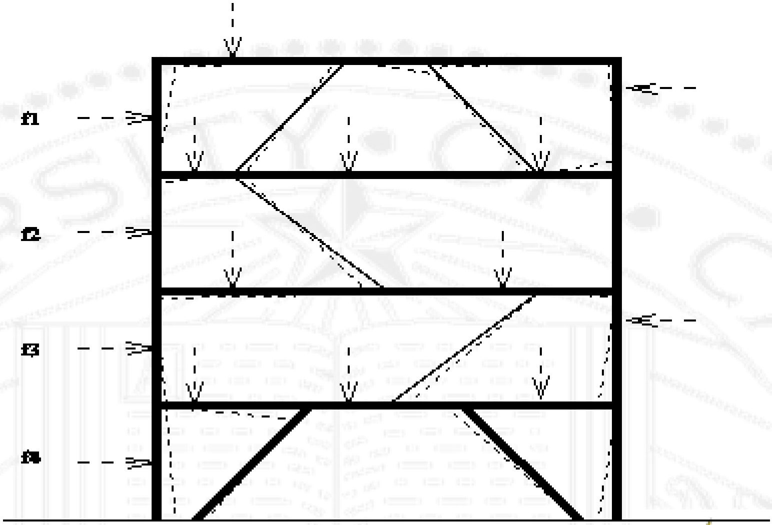
# Matrix Multiplication function

# Interpretations of *y=Ax*

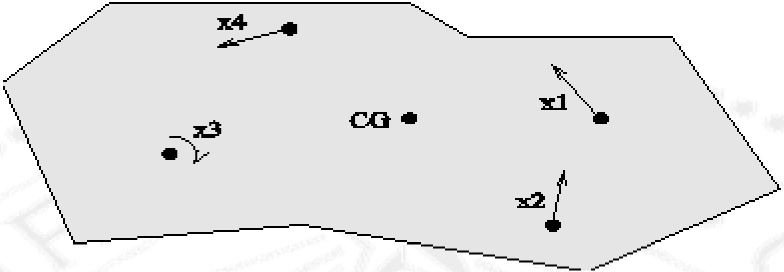
# Interpretations of $a_{ij}$

# Interpretations of $a_{ij}$

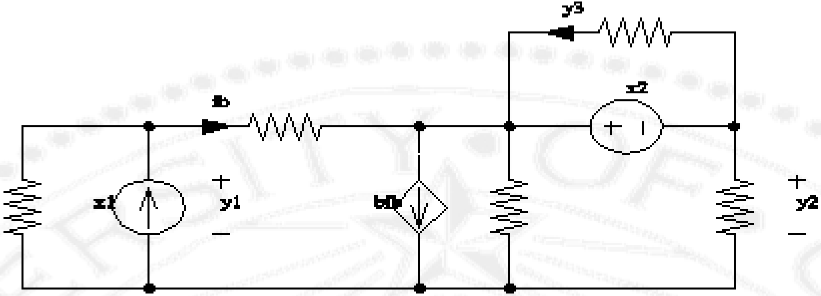
# Linear Elastic Structure



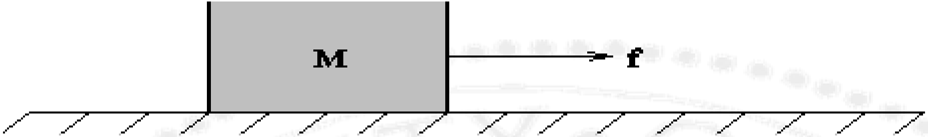
# Rigid Body Force/Torque



# Linear Static Circuit

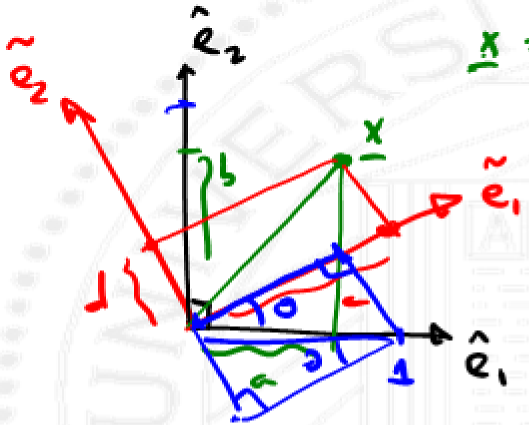


# Final Position/Velocity of Mass



# Questions?

$$f : \mathbb{R}^n \to \mathbb{R}^m \longleftrightarrow f(x) = \underbrace{A}_{\in\mathbb{R}^{m\times n}} \underbrace{x}_{\in \mathbb{R}^n}$$




$\underline{x} = a\hat{e}_1 + b\hat{e}_2$

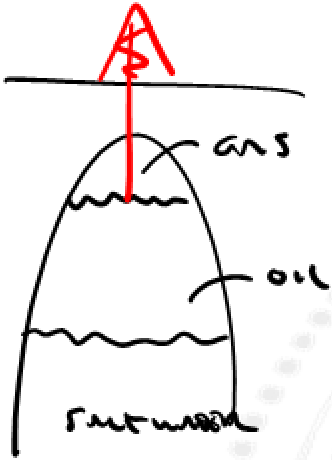
$y = c\tilde{e}_1 + d\tilde{e}_2$

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\-\sin\theta\end{bmatrix}$$

$\hat{e}_1$

# Graviometer Prospecting

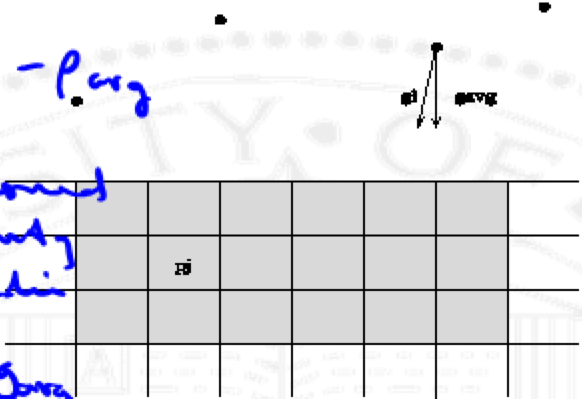
voxels




$x_j = \rho_j - \rho_{avg}$

$y_i$ = measured gravity gradient

$g_i - g_{avg}$




$y = Ax$

$x_j < 0$ salt water $10^{-7}$ g's

$x_j > 0$ uranium / mineral deposit

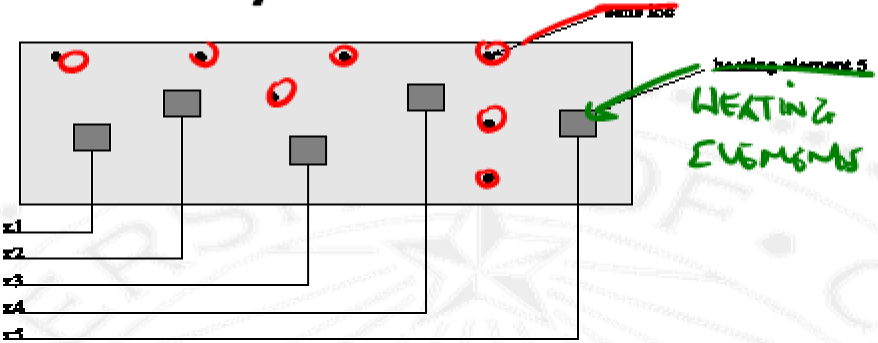
$x_j \ll 0$ gas

# Thermal System

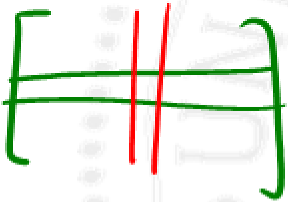
$x_j$ – power $j^{th}$ heating element

$y_i$: change in steady state temp @ location $i$

$y = Ax$

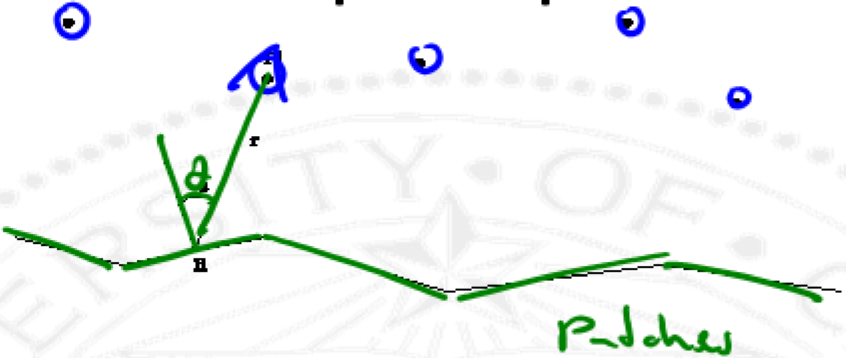



Diffusion Process / Poisson Eq'ns



# Illumination with Multiple Lamps

Lamps




n - lamps

m - small patches no shadows

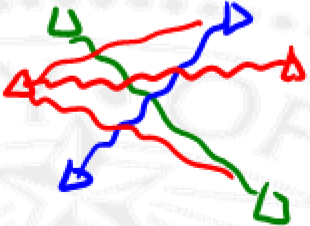
$x_j$ - power to $j^{th}$ lamp

$y_i$ - illumination on patch

$$a_{ij} = r_{ij}^{-2} \max(\cos\theta_{ij}, 0)$$

# Signal and Interference Power in Wireless System

n-transmitter/receiver pairs



$P_j$ – power of the $j^{th}$ transmitter

$s_i$ – received signal power @ $i^{th}$ receiver

$G_{ij}$ path gain from $j \to i$

$s_i / z_i \triangleq$ SNR

$s = Ap$

$A_{ij} = \begin{cases} G_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$z = Bp$

$B_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & \text{otherwise} \end{cases}$

# Cost of Production (1.2)

$y = Ax$

$x_j$: price per unit production input $j$

$y_i$: cost per unit of product $i$

$a_{ij}$: unit of production input required to manufacture one unit of product $i$

$q_i$: quantity of the product

$r_j$: total quantity of production input $j$ that is needed

# Cost of Production (2.2)

$r = A^T q$

$y = Ax$

$r^T x = (A^T q)^T x = q^T A x = q^T y$

# Network Traffic and Flows (1.2)

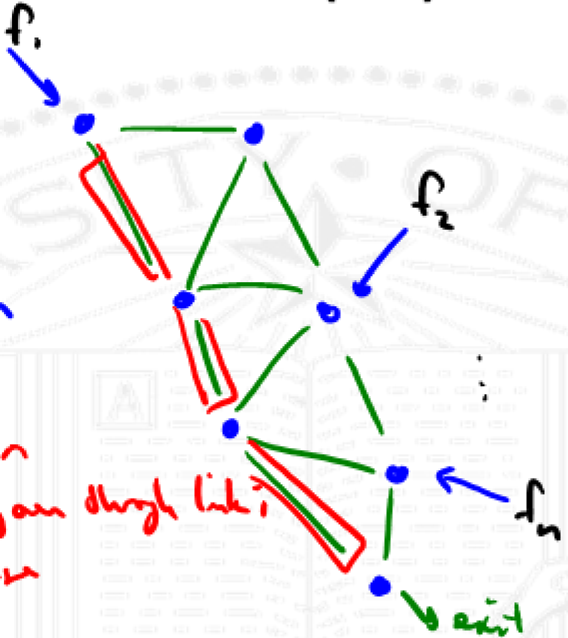
$n$ - flows w/ nodes $f_1 \cdots f_n$

$t_i$ - traffic on link $i$

$\Sigma$ flows that pass through link $i$

flow link incidence matrix

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes through link } i \\ 0 & \text{otherwise} \end{cases}$$



# Network Traffic and Flows (2.2)

$d_1 \dots d_n$ link delays

$l_1 \dots l_n$ latency (total time travel) flows

$t = Af$

the edges in the network

$$l = A^T d$$

$$f^T l = f^T A^T d = (Af)^T d = t^T d$$

$A^T$ has a very intuitive physical interpretation

# Linearization

$f: \mathbb{R}^n \to \mathbb{R}^m$ differentiable @ $x_0 \in \mathbb{R}^n$ then:

if $\underline{x}$ near $x_0$ $\longrightarrow$ $f(x)$ very near $f(x_0) + Df(x_0)(x - x_0)$

$$Df(x_0)_{ij} \triangleq \left.\frac{\partial f}{\partial x_j}\right|_{x=x_0} \leftarrow \text{Jacobian}$$

$y = f(x)$ ← non-lin

$y_0 = f(x_0)$

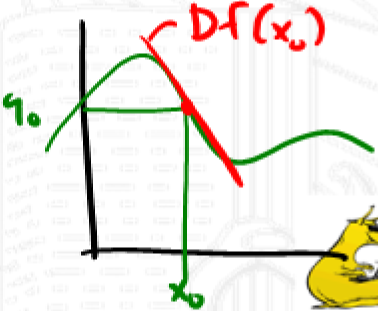
$\delta x \triangleq x - x_0$

$\delta y \triangleq y - y_0$

$$\underline{\delta y \simeq Df(x_0)\,\delta x}$$

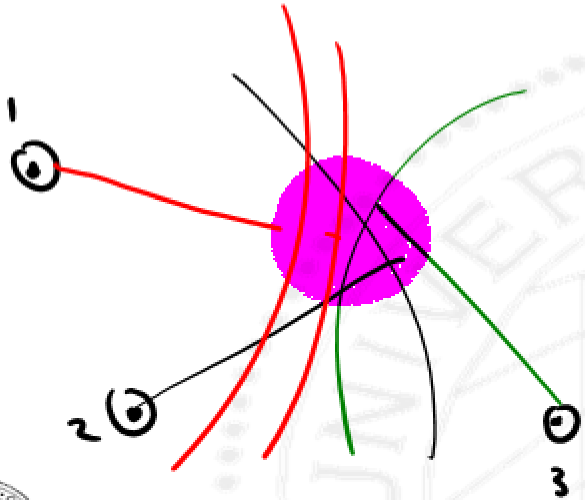
↑

unknown



# Navigation by Ranging (1.2)

$\rho_i$ distance from $i^{th}$ beacon



# Navigation by Ranging (2.2)

$\rho \in \mathbb{R}^4$ is a non-linear function of $(x, y) \in \mathbb{R}^2$

$(p, q)$

$\rho_i(x, y) = \sqrt{(x-p_i)^2 + (y-q_i)^2}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

$\delta\rho \cong A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$

$a_{i1} = \dfrac{x_0 - p_i}{\sqrt{(x_0-p_i)^2 + (y_0-q_i)^2}}$

$a_{i2} = \dfrac{y_0 - q_i}{\text{same}}$

# Broad Categories of Applications

- Linear model or function $y=Ax$
- Some broad categories of applications:
  - Estimation or inversion
  - Control or design
  - Mapping or transformation

(this list is not exclusive; can have combinations)

# Estimation or Inversion

$y_i$ is the $i^{th}$ measurement or sensor reading (which we know)

$x_j$ is a parameter to be estimated/determined

$a_{ij}$ sensitivity of $j^{th}$ parameter to $i^{th}$ sensor

→ find $x$, given $y$.

→ find all $x$'s that are consistent w/ $y$.

→ if no such $x$ exists, such that $y = Ax$,
find the best $x$ that is "most consistent" with $y$.

$y = Ax + v$ ← noise

# Control or Design

$x$ is a vector of design parameters or control inputs we can choose.

$y$ is the output or result

find $x$ such that $y \to y_{des}$

find all $x$'s that produce $y \to y_{des}$

among all such $x$'s that give $y \to y_{des}$

choose the <u>best</u> one.

# Mapping or Transformation

$x$ is mapped or transformed by $A$ into $y$

determine if there is an $x \to y$

find an $x$ that maps into $y$

find all $x$'s that map into $y$.

decode or undo transformation.

# Matrix Multiplication as Mixture of Columns

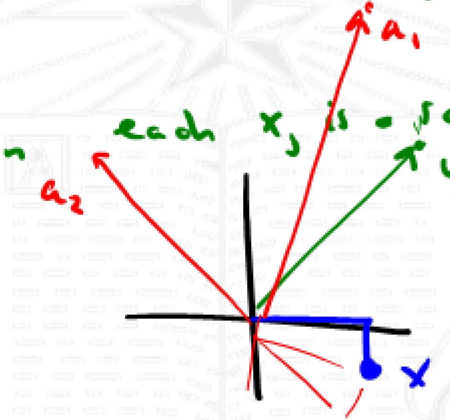
$y = Ax \qquad A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad a_j \in \mathbb{R}^m \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$y = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$ each $x_j$ is a scalar

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$y = Ax = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$



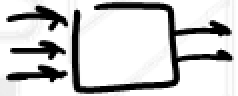
# Unit Vectors

$x = e_j$ th $j^{th}$ unit vector

$$e_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad e_n \triangleq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

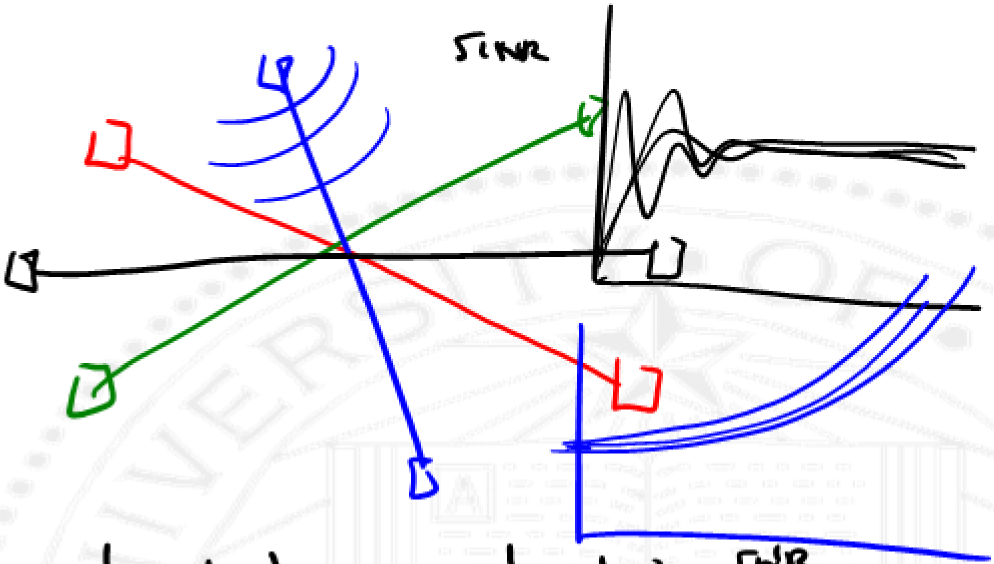
$A e_j = a_j \leftarrow j^{th}$ column of $A$

inputs



outputs

## Questions




algorithm to adjust power based on SINR

$\gamma$

# Matrix multiplication as inner product with rows

$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_n^T \end{bmatrix}$

row ← gain from output to all inputs

$y = Ax = \begin{bmatrix} \tilde{a}_1^T x \\ \tilde{a}_2^T x \\ \vdots \\ \tilde{a}_n^T x \end{bmatrix}$ $\quad \langle \hat{a}_i , x \rangle$

inner product

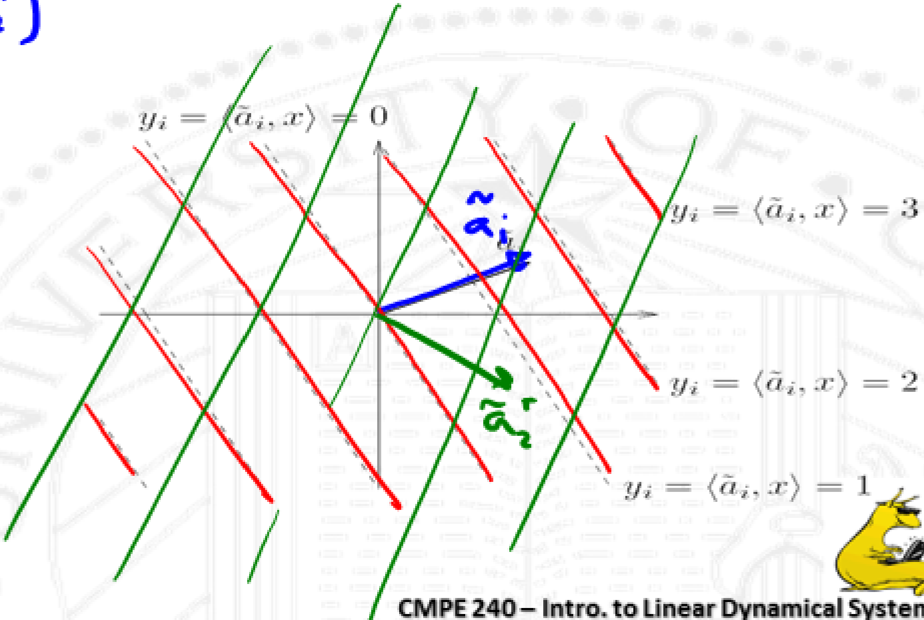
$y_i$ inner product of $\langle \tilde{a}_i , x \rangle$

$y_i = \tilde{a}_i^T x = \langle \tilde{a}_i , x \rangle = \alpha$ in $\mathbb{R}^n$

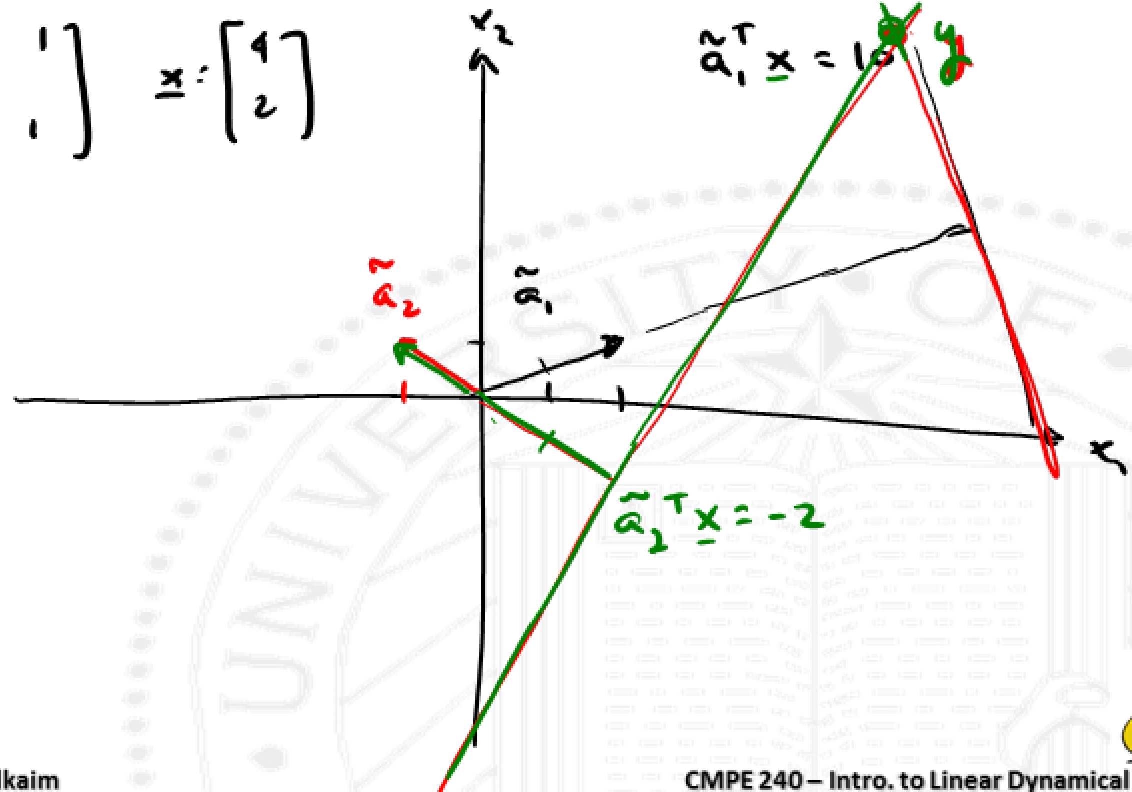
# Geometric Interpretation

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \quad x = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\tilde{a}_i^T x = \alpha$$



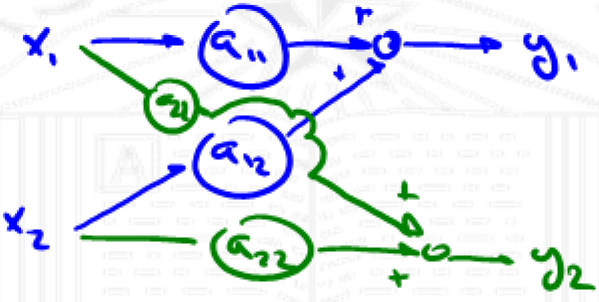
$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



# Block Diagram Representation

$y = Ax$ signal flow graph or block diagram

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$




Full Bipartite graph

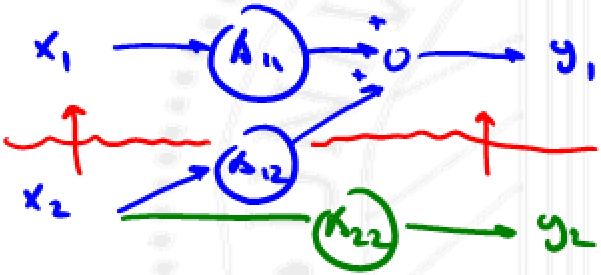
# Example: Block Upper Triangular

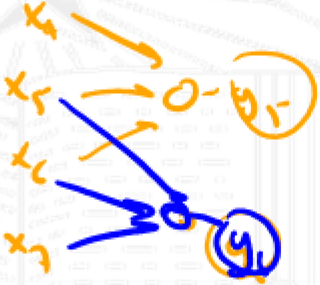
$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

$$A_{11} \in \mathbb{R}^{m_1 \times n_1}$$

$$y_1 = A_{11} x_1 + A_{12} x_2$$

$$y_2 = A_{22} x_2$$



# Matrix Multiplication as Composition

inner dimensions must agree

$A \in \mathbb{R}^{m \times n} \qquad B \in \mathbb{R}^{n \times p} \qquad C \triangleq AB \in \mathbb{R}^{m \times p}$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \sim O(n^3) \sim k\,O(n^{1.2})$$

$y = Cz \longrightarrow y = Ax \quad x = Bz \qquad Cz = A(Bz)$

$$z \xrightarrow[p]{} \boxed{B} \xrightarrow[n]{} \boxed{A} \xrightarrow[m]{} y$$

# Column and Row Interpretations

$$C = AB = [Ab_1 \; Ab_2 \cdots Ab_p]$$

$$C = AB = \begin{bmatrix} \tilde{a}_1^T B \\ \tilde{a}_2^T B \\ \vdots \\ \tilde{a}_n^T B \end{bmatrix}$$

parallel vector/matrix mult. problem

# Inner Product Interpretation

$$c_{ij} = \tilde{a}_i^T b_j = \langle \tilde{a}_i , b_j \rangle$$

$c_{ij} = \phi$ $i^{th}$ row of A $\perp$ $j^{th}$ colmn of B

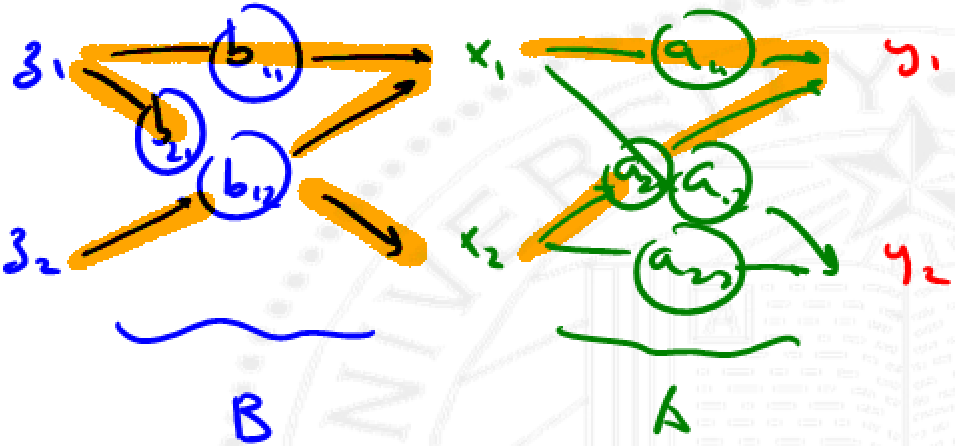
Gram matrix $f_1 \cdots f_n$ $G_{ij} = f_i^T f_j$

$$G = [f_1 \cdots f_n]^T [f_1 \cdots f_n]$$

$$AB = I \longrightarrow \tilde{a}_i^T b_j = \begin{cases} 1 & i = j \\ \phi & \text{otherwise} \end{cases}$$

$y = Cz \longleftrightarrow y = Ax \quad x = Bz$

# Matrix Multiplication Interpretation via Paths




$C_{ij}$ total path gain from input $j$ to output $i$

# Questions?