



PROBLEM SET 1

1. *A simple power control algorithm for a wireless network.* First some background. We consider a network of n transmitter/receiver pairs. Transmitter i transmits at power level p_i (which is positive). The path gain from transmitter j to receiver i is G_{ij} (which are all nonnegative, and G_{ii} are positive).

The signal power at receiver i is given by $s_i = G_{ii}p_i$. The noise plus interference power at receiver i is given by

$$q_i = \sigma + \sum_{j \neq i} G_{ij}p_j$$

where $\sigma > 0$ is the self-noise power of the receivers (assumed to be the same for all receivers). The *signal to interference plus noise ratio* (SINR) at receiver i is defined as $S_i = s_i/q_i$.

For signal reception to occur, the SINR must exceed some threshold value γ (which is often in the range 3 – 10). Various *power control algorithms* are used to adjust the powers p_i to ensure that $S_i \geq \gamma$ (so that each receiver can receive the signal transmitted by its associated transmitter). In this problem, we consider a simple power control update algorithm.

The powers are all updated synchronously at a fixed time interval, denoted by $t = 0, 1, 2, \dots$. Thus the quantities p , q , and S are discrete-time signals, so for example $p_3(5)$ denotes the transmit power of transmitter 3 at time epoch $t = 5$. What we'd like is

$$S_i(t) = s_i(t)/q_i(t) = \alpha\gamma$$

where $\alpha > 1$ is an SINR safety margin (of, for example, one or two dB). Note that increasing $p_i(t)$ (power of the i th transmitter) increases S_i but decreases all other S_j .

A very simple power update algorithm is given by

$$p_i(t+1) = p_i(t)(\alpha\gamma/S_i(t)). \tag{1}$$

This scales the power at the next time step to be the power that would achieve $S_i = \alpha\gamma$, if the interference plus noise term were to stay the same. But unfortunately, changing the transmit powers also changes the interference powers, so it's not that simple!

Finally, we get to the problem.

- (a) Show that the power control algorithm (1) can be expressed as a linear dynamical system with constant input, *i.e.*, in the form

$$p(t+1) = Ap(t) + b,$$

where $A \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$ are constant. Describe A and b explicitly in terms of σ, γ, α and the components of G .

- (b) *Matlab simulation.* Use matlab to simulate the power control algorithm (1), starting from various initial (positive) power levels. Use the problem data

$$G = \begin{bmatrix} 1 & .2 & .1 \\ .1 & 2 & .1 \\ .3 & .1 & 3 \end{bmatrix}, \quad \gamma = 3, \quad \alpha = 1.2, \quad \sigma = 0.01.$$

Plot S_i and p as a function of t , and compare it to the target value $\alpha\gamma$. Repeat for $\gamma = 5$. Comment briefly on what you observe.

Comment: You'll soon understand what you see.

2. *State equations for a linear mechanical system.* The equations of motion of a lumped mechanical system undergoing small motions can be expressed as

$$M\ddot{q} + D\dot{q} + Kq = f$$

where $q(t) \in \mathbf{R}^k$ is the vector of deflections, M , D , and K are the *mass*, *damping*, and *stiffness* matrices, respectively, and $f(t) \in \mathbf{R}^k$ is the vector of externally applied forces. Assuming M is invertible, write linear system equations for the mechanical system, with state $x = [q^T \ \dot{q}^T]^T$, input $u = f$, and output $y = q$.

3. *Representing linear functions as matrix multiplication.* Suppose that $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is linear. Show that there is a matrix $A \in \mathbf{R}^{m \times n}$ such that for all $x \in \mathbf{R}^n$, $f(x) = Ax$. (Explicitly describe how you get the coefficients A_{ij} from f , and then verify that $f(x) = Ax$ for any $x \in \mathbf{R}^n$.)

Is the matrix A that represents f unique? In other words, if $\tilde{A} \in \mathbf{R}^{m \times n}$ is another matrix such that $f(x) = \tilde{A}x$ for all $x \in \mathbf{R}^n$, then do we have $\tilde{A} = A$? Either show that this is so, or give an explicit counterexample.

4. *Some linear functions associated with a convolution system.* Suppose that u and y are scalar-valued discrete-time signals (*i.e.*, sequences) related via convolution:

$$y(k) = \sum_j h_j u(k-j), \quad k \in \mathbf{Z},$$

where $h_k \in \mathbf{R}$. You can assume that the convolution is *causal*, *i.e.*, $h_j = 0$ when $j < 0$.

- (a) *The input/output (Toeplitz) matrix.* Assume that $u(k) = 0$ for $k < 0$, and define

$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix}, \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}.$$

Thus U and Y are vectors that give the first $N + 1$ values of the input and output signals, respectively. Find the matrix T such that $Y = TU$. The matrix T describes the linear mapping from (a chunk of) the input to (a chunk of) the output. T is called the input/output or Toeplitz matrix (of size $N + 1$) associated with the convolution system.

(b) *The Hankel matrix.* Now assume that $u(k) = 0$ for $k > 0$ or $k < -N$ and let

$$U = \begin{bmatrix} u(0) \\ u(-1) \\ \vdots \\ u(-N) \end{bmatrix}, \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}.$$

Here U gives the *past input* to the system, and Y gives (a chunk of) the resulting future output. Find the matrix H such that $Y = HU$. H is called the Hankel matrix (of size $N + 1$) associated with the convolution system.

5. *Matrix representation of polynomial differentiation.* We can represent a polynomial of degree less than n ,

$$p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0,$$

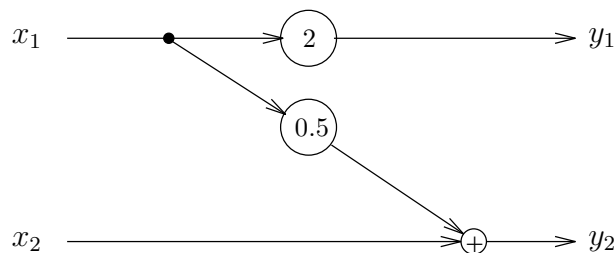
as the vector $[a_0 \ a_1 \ \cdots \ a_{n-1}]^T \in \mathbf{R}^n$. Consider the linear transformation \mathcal{D} that differentiates polynomials, *i.e.*, $\mathcal{D}p = dp/dx$. Find the matrix D that represents \mathcal{D} (*i.e.*, if the coefficients of p are given by a , then the coefficients of dp/dx are given by Da).

6. *Some sparsity patterns.*

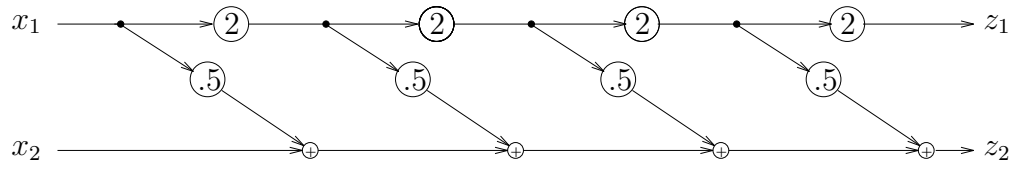
- (a) A matrix $A \in \mathbf{R}^{n \times n}$ is *tridiagonal* if $A_{ij} = 0$ for $|i - j| > 1$. Draw a block diagram of $y = Ax$ for A tridiagonal.
- (b) Consider a certain linear mapping $y = Ax$ with $A \in \mathbf{R}^{m \times n}$. For i odd, y_i depends only on x_j for j even. Similarly, for i even, y_i depends only on x_j for j odd. Describe the sparsity structure of A . Give the structure a reasonable, suggestive name.

7. *Matrices and signal flow graphs.*

(a) Find $A \in \mathbf{R}^{2 \times 2}$ such that $y = Ax$ in the system below:



(b) Find $B \in \mathbf{R}^{2 \times 2}$ such that $z = Bx$ in the system below:



Do this two ways: first, by expressing the matrix B in terms of A from the previous part (explaining why they are related as you claim); and second, by directly evaluating all possible paths from each x_j to each z_i .