5.6 Nonlinear stability of Rayleigh-Bénard convection

Linear stability only tells us whether a system is stable or unstable to infinitesimally-small perturbations. Weakly nonlinear theory only pushes this formal analysis further by an amount "ϵ" into the nonlinear domain, and truncated systems are not reliable since they discard dynamics that may turn out to be important. However, we know from everyday life, dynamical systems theory, laboratory experiments, and numerical experiments, that there are many cases in which a system can be unstable to finite-amplitude perturbations (i.e. perturbations of sufficiently large amplitude) even if it is linearly or weakly nonlinearly stable. There are very few tools to study the general nonlinear stability of such a system, but one of them is called energy stability theory. Energy stability is akin to Lyapunov stability in dynamical systems theory, so we will begin by recalling how Lyapunov stability is defined.

5.6.1 Lyapunov stability in dynamical systems

Lyapunov stability theory is an excellent way of proving whether a steady state is globally stable instead of being just linearly stable. To see how it works, it is best to start with a simple example based on a 2D dynamical system.

Consider the following system:

$$\begin{aligned} \dot{f} &= -f + 4g \\ \dot{g} &= -f - g^3 \end{aligned} \tag{5.129}$$

It has an obvious fixed point at $f = g = 0$. Linearizing around it, we find that small perturbations satisfy

$$\begin{aligned} \dot{f} &= -f + 4g \\ \dot{g} &= -f \end{aligned} \tag{5.130}$$

so

$$\ddot{f} = -\dot{f} - 4f \tag{5.131}$$

This suggests that $f \propto e^{\lambda t}$ with $\lambda^2 + \lambda + 4 = 0$. This has solutions

$$\lambda = \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{15}}{2} \tag{5.132}$$

so that

$$f(t) = e^{-t/2}\left(a\cos(\sqrt{15}t/2) + b\sin(\sqrt{15}t/2)\right) \tag{5.133}$$

where a and b are two integration constants, and similarly for g. This implies that, linearly speaking at least, the origin is a stable spiral. But this is only true for initial conditions close to the origin. Do *all* possibly initial conditions always end up decaying to 0 as well?

To answer this question, let's construct a *Lyapunov function* $E(t)$. By definition, a Lyapunov function has to be strictly positive, must be equal to zero at the fixed point (here, $f = g = 0$), and has satisfy $dE/dt < 0$ except at the fixed point, where it must be 0. Let's try:

$$E(t) = f^2 + \gamma^2 g^2 \tag{5.134}$$

where γ^2 remains to be determined (but is positive). By construction, we see that E is indeed positive everywhere except at $f = g = 0$ where it is 0. Furthermore,

$$\frac{dE}{dt} = 2(f\dot{f} + \gamma^2 g\dot{g}) = 2(-f^2 + 4gf - \gamma^2 gf - \gamma^2 g^4) \tag{5.135}$$

If we take $\gamma^2 = 4$, then the term in fg conveniently vanishes, and we are left with

$$\frac{dE}{dt} = -2f^2 - 8g^4 \tag{5.136}$$

which is clearly negative, except at the fixed point.

What does this buy us? Well, we see that given *any* initial condition f_0, g_0, the dynamical system will evolve in time following (5.130). However, as we have just demonstrated, this *also* means that E will evolve in time according to (5.136), and will therefore *always decrease* since $dE/dt < 0$. As $t \to \infty$, $E \to 0$ (since E has to be positive), which then necessarily implies that both f and g must also be going to 0.

To summarize, it is possible to prove that a fixed point is globally stable provided we can find a scalar function E of the *dependent* variables, that satisfies:

- E is strictly positive, except at the fixed point where it must be 0
- dE/dt is strictly negative, except at the fixed point where it must be 0

A nice feature of this method is that it very easily generalizes to systems with any number of dimensions, and can be used in fluid dynamics to prove the global stability of a steady state. As we will now demonstrate below using very similar arguments, convection is an example of a system where linear stability also implies global stability. Of course, not all steady states that are linearly stable are also globally stable. In fact, it is the opposite situation that can be interesting, and we will see some examples of that alternative type of dynamics later in the course.

5.6.2 Energy stability of Rayleigh Bénard convection

We now generalize the idea of Lyapunov stability to PDEs (notably, the Rayleigh Bénard problem) and attempt to create a Lyapunov function to study the stability properties of convection. Since E has to be a scalar function, and yet has to capture the dynamics of the *whole* fluid system, it is best to create is as an integral over a domain D, where we take D to be the space between the plates.

Since this has infinite horizontal extent, we then reduce it to some portion of the horizontal plane, and require periodicity in x (recall that we are considering here a 2D problem only). Hence

$$E = \langle \text{ stuff } \rangle \tag{5.137}$$

where $\langle \cdot \rangle$ denotes the spatial integral over D.

For reasons that will be apparent shortly, it is also best to make E quadratic in the dependent variables, rather than, say, quartic, or higher-order. The simplest quadratic, positive definite integral we know is the one that is based, for instance, on the kinetic energy of the fluid. Dotting the momentum equation with $\boldsymbol{u}$, and integrating over a domain, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle |\boldsymbol{u}|^2 \rangle + \frac{1}{2}\langle \boldsymbol{u} \cdot \nabla |\boldsymbol{u}|^2 \rangle = -\langle \boldsymbol{u} \cdot \nabla p \rangle + \langle \mathrm{RaPr} wT \rangle + \langle \mathrm{Pr} \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \rangle \tag{5.138}$$

Since $\nabla \cdot \boldsymbol{u} = 0$, we have

$$\boldsymbol{u} \cdot \nabla |\boldsymbol{u}|^2 = \nabla \cdot (\boldsymbol{u}|\boldsymbol{u}|^2) \text{ and } \boldsymbol{u} \cdot \nabla p = \nabla \cdot (p\boldsymbol{u}) \tag{5.139}$$

Furthermore, since the boundary conditions are $w = 0$ on the top and bottom boundary, and periodic in x, the integral over the domain of these divergences are all zero. Finally, using integration by parts and the same properties of the boundary conditions, we have (using Einstein's convention of repeated indices)

$$\langle \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \rangle = \langle u_i \partial_{jj} u_i \rangle = -\langle (\partial_j u_i)^2 \rangle = -\langle |\nabla \boldsymbol{u}|^2 \rangle \tag{5.140}$$

The kinetic energy equation then becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\langle |\boldsymbol{u}|^2 \rangle = \mathrm{RaPr}\langle wT \rangle - \mathrm{Pr}\langle |\nabla \boldsymbol{u}|^2 \rangle \tag{5.141}$$

This states that the total kinetic energy in the domain changes as a results of the conversion of potential energy (first terms on the RHS) or viscous dissipation (second term on the RHS). While viscous dissipation is always negative, the first term can be positive (and must be, for instability to occur!). If that is the case, $\langle |\boldsymbol{u}|^2 \rangle$ can either increase or decay depending on which of the two terms, energy injection or energy dissipation, is the largest.

A similar evolution equation for another positive definite functional can be constructed by considering the thermal energy equation instead, and multiplying it by T. Integrating over the same domain D, using the same trick to get rid of the divergence, and integrating the thermal diffusion term by parts, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle T^2 \rangle = \langle wT \rangle - \langle |\nabla T|^2 \rangle \tag{5.142}$$

Again, we see that $\langle T^2 \rangle$ can either increase or decay depending on the relative sizes of the first and second term on the RHS.

We can now construct a very general quadratic Lyapunov functional as $E(\boldsymbol{u},T) = (1/2)\langle|\boldsymbol{u}|^2 + \gamma^2 T^2\rangle$, where γ^2 is an arbitrary positive constant. The evolution equation for E is then

$$\frac{\partial E}{\partial t} = (\mathrm{RaPr} + \gamma^2)\langle wT\rangle - \mathrm{Pr}\langle|\nabla\boldsymbol{u}|^2\rangle - \gamma^2\langle|\nabla T|^2\rangle \tag{5.143}$$

If we can somehow prove that, for all non-zero functions u, w and T (satisfying $\nabla\cdot\boldsymbol{u} = 0$) the RHS of this equation is *strictly* negative except at the fixed point, then E *must strictly decrease with time.* Since $E \geq 0$, the only possible evolution of this system drives E towards 0, so that $E \to 0$ as $t \to \infty$. In other words, *all perturbations must decay*, and the system is globally stable. Given that this proof uses an *energy-like* functional to show global stability, the criterion derived is often called *energy stability*.

In order to determine when $dE/dt < 0$, it is sufficient to show that $(\mathrm{RaPr} + \gamma^2)\langle wT\rangle$ is smaller than the dissipation term $\mathcal{D} = \mathrm{Pr}\langle|\nabla\boldsymbol{u}|^2\rangle + \gamma^2\langle|\nabla T|^2\rangle$ for all possible functions u, w, T (satisfying $\nabla\cdot\boldsymbol{u} = 0$). To do that, we now fix the total dissipation, and maximize $(\mathrm{RaPr}+\gamma^2)\langle wT\rangle$, subject to the constraints $\mathcal{D} = D_0$ (where D_0 is known), and $\nabla\cdot\boldsymbol{u} = 0$. Energy stability would then simply require that this maximum value be smaller than D_0.

In order to maximize $(\mathrm{RaPr}+\gamma^2)\langle wT\rangle$ subject to these condition, we introduce the Lagrange multipliers Λ_1 and Λ_2 , and maximize instead

$$\mathcal{S} = (\mathrm{RaPr} + \gamma^2)\langle wT\rangle + \langle\Lambda_1(\mathrm{Pr}|\nabla\boldsymbol{u}|^2 + \gamma^2|\nabla T|^2 - D_0)\rangle + \langle\Lambda_2\nabla\cdot\boldsymbol{u}\rangle \tag{5.144}$$

over all functions u, w, T, *and* Λ_2. Note how each Lagrange multiplier is associated with one of the constraints. While Λ_1 is a constant, because we are trying to impose $\mathcal{D} = D_0$ globally, Λ_2 is a function because we want to enforce $\nabla\cdot\boldsymbol{u}$ at every point in the domain D. We are now simply left to maximize $\mathcal{S}$.

Optimization using Euler-Lagrange equations.

Let's recall how one may go about maximizing a functional (rather than a function). Consider the much simpler functional, say,

$$\mathcal{S}(f) = \int_a^b \mathcal{L}(f,\dot{f};x)dx \tag{5.145}$$

where $\dot{f} = df/dx$ and where f is subject to simple conditions such as $f(a) = f_a$ and $f(b) = f_b$.

Stating that f is the function that maximizes $\mathcal{S}$ is equivalent to saying that infinitesimal variations in f result in a zero change in $\mathcal{S}$, at least at first order. Indeed, near the maximum $x_{\max}$ of a normal single-variable function $g(x)$,

$$g(x) = g(x_{\max}) + 0.5(x-x_{\max})^2 g''(x_{\max}) \to g(x) - g(x_{\max}) \simeq 0 + O((x-x_{\max})^2) \tag{5.146}$$

The same is true for $\mathcal{S}$, so if $f(x) = f_{\max}(x) + \delta f(x)$, where $f_{\max}(x)$ is the function which maximizes $\mathcal{S}$, and $\delta f(x)$ is a small perturbation around it, then we expect that

$$\delta\mathcal{S} = \mathcal{S}(f_{\max} + \delta f) - \mathcal{S}(f_{\max}) \simeq 0 \tag{5.147}$$

This condition is the one that effectively yields $f_{\max}$.

Indeed, let's evaluate $\delta\mathcal{S}$:

$$\delta\mathcal{S} = \int_a^b \left[\mathcal{L}(f_{\max} + \delta f, \dot{f}_{\max} + \delta\dot{f}; x) - \mathcal{L}(f_{\max}, \dot{f}_{\max}; x)\right] dx \equiv \int_a^b \delta\mathcal{L} dx \tag{5.148}$$

which defines $\delta\mathcal{L}$. Since

$$\delta\mathcal{L} = \delta f \frac{\partial\mathcal{L}}{\partial f} + \delta\dot{f}\frac{\partial\mathcal{L}}{\partial\dot{f}} \tag{5.149}$$

then

$$\delta\mathcal{S} = \int_a^b \left[\delta f \frac{\partial\mathcal{L}}{\partial f} + \delta\dot{f}\frac{\partial\mathcal{L}}{\partial\dot{f}}\right] dx \tag{5.150}$$

Finally, note that $\delta\dot{f} = d(\delta f)/dx$ so, using integration by parts,

$$\int_a^b \frac{\partial\mathcal{L}}{\partial\dot{f}}\frac{d\delta f}{dx}dx = \left[\frac{\partial\mathcal{L}}{\partial\dot{f}}\delta f\right]_a^b - \int_a^b \delta f \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\dot{f}}dx \tag{5.151}$$

Since f has to satisfy the boundary conditions, we cannot perturb it at $x = a$ and $x = b$. This means that $\delta f(a) = \delta f(b) = 0$, so the integrated term is equal to 0. This leaves us with:

$$\delta\mathcal{S} = \int_a^b \left[\delta f \frac{\partial\mathcal{L}}{\partial f} - \delta f \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\dot{f}}\right] dx = \int_a^b \delta f \left[\frac{\partial\mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\dot{f}}\right] dx = 0 \tag{5.152}$$

For this to be true for any possible perturbing function $\delta f(x)$, the term in the square brackets have to be zero. In other words, the function $f_{\max}$ satisfies the equation

$$\frac{\partial\mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\dot{f}} = 0 \tag{5.153}$$

(with the boundary condition $f(a) = f_a$, and $f(b) = f_b$). This equation is called an Euler-Lagrange equation.

Note that this method can easily be generalized when $\mathcal{L}$ is a functional of many dependent variables $\{f_i\}_{i=1..I}$ and when the integral is in many dimensions $\{x_j\}_{j=1..J}$. For each f_i, we have

$$\frac{\partial\mathcal{L}}{\partial f_i} - \sum_j \frac{\partial}{\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial f_i/\partial x_j)} = 0 \tag{5.154}$$

Condition for energy stability.

We now use Euler-Lagrange's equations to maximize $\mathcal{S}$ given by (5.144). Using the notation of the previous section, $\mathcal{S}$ is given by

$$\mathcal{S} = \int \mathcal{L} dx dz \tag{5.155}$$

where $\mathcal{L}$ is the functional

$$\mathcal{L} = (\mathrm{RaPr} + \gamma^2) wT + \Lambda_1(\mathrm{Pr}|\nabla \boldsymbol{u}|^2 + \gamma^2 |\nabla T|^2 - D_0) + \Lambda_2 \nabla \cdot \boldsymbol{u} \tag{5.156}$$

where, recall, Λ_1 is a constant while Λ_2 is a function of x and z. Since we have two independent variables, we have to calculate

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial q/\partial x)} - \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial(\partial q/\partial z)} = 0 \tag{5.157}$$

where q is either u, w, T or Λ_2.

Let's work first with the derivative with respect to Λ_2, which is the simplest one since $\mathcal{L}$ does not depend on any *derivatives* of Λ_2. We simply have

$$\frac{\partial \mathcal{L}}{\partial \Lambda_2} = \nabla \cdot \boldsymbol{u} = 0 \tag{5.158}$$

which recovers the incompressibility constraint. This suggests that, as usual, we can represent $\boldsymbol{u}$ by using a stream function with $u = \partial\phi/\partial z$ and $w = -\partial\phi/\partial x$. Similarly, the derivative with respect to Λ_1 also just recovers the constraint $\mathcal{D} = D_0$.

Let's now work with the derivative with respect to T. We have

$$\frac{\partial \mathcal{L}}{\partial T} = (\mathrm{RaPr} + \gamma^2) w \tag{5.159}$$

while

$$\frac{\partial \mathcal{L}}{\partial(\partial T/\partial x)} = 2\gamma^2 \Lambda_1 \frac{\partial T}{\partial x} \tag{5.160}$$

since

$$|\nabla T|^2 = \left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2 \tag{5.161}$$

and similarly for the derivative with respect to $\partial T/\partial z$. Putting these together using (5.157), we then get

$$(\mathrm{RaPr} + \gamma^2) w - 2\gamma^2 \Lambda_1 \nabla^2 T = 0 \tag{5.162}$$

Similarly, it can be shown that

$$\begin{aligned} (\mathrm{RaPr} + \gamma^2) T - \frac{\partial \Lambda_2}{\partial z} - 2\mathrm{Pr}\Lambda_1 \nabla^2 w &= 0 \\ -\frac{\partial \Lambda_2}{\partial x} - 2\mathrm{Pr}\Lambda_1 \nabla^2 u &= 0 \end{aligned} \tag{5.163}$$

We can then eliminate Λ_2 between the two momentum-like equations, to get

$$(\mathrm{RaPr} + \gamma^2)\frac{\partial T}{\partial x} = -2\mathrm{Pr}\Lambda_1 \nabla^4 \phi \tag{5.164}$$

and finally we can eliminate, say, T, to get

$$(\mathrm{RaPr} + \gamma^2)^2 \frac{\partial^2 \phi}{\partial x^2} = 4\mathrm{Pr}\gamma^2 \Lambda_1^2 \nabla^6 \phi \tag{5.165}$$

This shows that the solution ϕ that maximizes the functional $\mathcal{S}$ is the solution of a linear eigenvalue problem, where the eigenvalue is Λ_1 (all the other parameters being known and fixed). Since the solutions have to satisfy the same boundary conditions as the original problem (i.e. periodic in x and impermeable, stress-free in z, with T given on the boundaries), they have to be of the form

$$\begin{aligned} \phi(x,z) &= \hat{\phi} e^{ik_x x} \sin(n\pi z) \\ ik_x T(x,z) &= -\frac{2\mathrm{Pr}\Lambda_1}{(\mathrm{RaPr} + \gamma^2)}(k_x^2 + n^2\pi^2)^2 \phi(x,z) \end{aligned} \tag{5.166}$$

(where we implicitly mean the real part of these quantities) with

$$k_x^2 (\mathrm{RaPr} + \gamma^2)^2 = 4\mathrm{Pr}\gamma^2 \Lambda_1^2 (k_x^2 + n^2\pi^2)^3 \tag{5.167}$$

Let's now go back the original question, and determine under which condition the maximum of $(\mathrm{RaPr} + \gamma^2)\langle wT \rangle$ is indeed smaller than D_0. First note that by (5.162), *for the optimal functions*,

$$(\mathrm{RaPr} + \gamma^2)\langle wT \rangle = 2\gamma^2 \Lambda_1 \langle T \nabla^2 T \rangle = -2\gamma^2 \Lambda_1 \langle |\nabla T|^2 \rangle \tag{5.168}$$

We are then left to estimate the sign of

$$\frac{dE}{dt} = (-2\Lambda_1 - 1)\gamma^2 \langle |\nabla T|^2 \rangle - \mathrm{Pr}\langle |\nabla \boldsymbol{u}|^2 \rangle \tag{5.169}$$

Using (5.166), we have

$$\langle |\nabla T|^2 \rangle = (k_x^2 + n^2\pi^2)\langle T^2 \rangle = \frac{4\mathrm{Pr}^2 \Lambda_1^2}{k_x^2 (\mathrm{RaPr} + \gamma^2)^2}(k_x^2 + n^2\pi^2)^5 \langle \phi^2 \rangle \tag{5.170}$$

while

$$\begin{aligned} \langle |\nabla \boldsymbol{u}|^2 \rangle &= \langle \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial x \partial z}\right)^2 + \left(\frac{\partial^2 \phi}{\partial z^2}\right)^2 \rangle \\ &= (k_x^2 + n^2\pi^2)^2 \langle \phi^2 \rangle \end{aligned} \tag{5.171}$$

so

$$\frac{dE}{dt} = \left[(-2\Lambda_1 - 1)\gamma^2 \frac{4\mathrm{Pr}\Lambda_1^2}{k_x^2(\mathrm{RaPr} + \gamma^2)^2}(k_x^2 + n^2\pi^2)^3 - 1\right] \mathrm{Pr}(k_x^2 + n^2\pi^2)^2 \langle \phi^2 \rangle \tag{5.172}$$

We can simplify this greatly using (5.167):

$$\frac{dE}{dt} = -(2\Lambda_1 + 2)\mathrm{Pr}(k_x^2 + n^2\pi^2)^2\langle\phi^2\rangle \tag{5.173}$$

which is always negative as long as $2\Lambda + 2 > 0$, which implies $\Lambda_1 > -1$.

Recall that Λ_1 is the solution of (5.167), so

$$\Lambda_1 = \pm\frac{k_x(\mathrm{RaPr} + \gamma^2)}{2\sqrt{\mathrm{Pr}}\gamma(k_x^2 + n^2\pi^2)^{3/2}} \tag{5.174}$$

Note that if $\Lambda_1 > 0$, energy stability is always guaranteed because of (5.169). The interval we need to worry about is therefore $-1 < \Lambda_1 < 0$. The condition for the *negative* root to be larger than -1 is equivalent to saying that

$$\frac{(\mathrm{RaPr} + \gamma^2)^2}{4\mathrm{Pr}\gamma^2} < \frac{(k_x^2 + n^2\pi^2)^3}{k_x^2} \tag{5.175}$$

This will always be true as long as the LHS of this inequality is smaller than any possible value that the RHS may take. As it turns out, we have already worked out the minimum of this expression – it's the same as in linear theory! The minimum value, $27\pi^4/4$, is achieved for $n = 1$, and for $k_x^2 = \pi^2/2$. Energy stability is therefore guaranteed provided:

$$(\mathrm{RaPr} + \gamma^2)^2 < 27\mathrm{Pr}\pi^4\gamma^2 \tag{5.176}$$

At this point, it is worth recalling that we constructed not a single Lyapunov function, but an entire family of them – each corresponding to a different value of γ. For each Lyapunov function, we get a sufficient criterion for energy stability as $\mathrm{Ra} < \mathrm{Ra_c}(\gamma)$ where

$$\mathrm{Ra}_c(\gamma) = \frac{\sqrt{27\mathrm{Pr}}\pi^2\gamma - \gamma^2}{\mathrm{Pr}} \tag{5.177}$$

To find the maximum possible value of Ra below which it is possible to guarantee stability, we simply have to choose the γ that maximizes the RHS of this last inequality. This occurs when $\gamma = \sqrt{27\mathrm{Pr}}\pi^2/2$. Putting everything together, we can then prove the following result: if

$$\mathrm{Ra} < \max_\gamma \mathrm{Ra_c}(\gamma) = \frac{27\pi^4}{4} \tag{5.178}$$

then the system is *energy stable*. Note how this critical value is *exactly the same* as the one we had obtained for the linear stability criterion.

This rather remarkable result proves that the criterion for linear stability in Rayleigh-Bénard convection is *also* the criterion for global stability. This implies that below $\mathrm{Ra}_c = 27\pi^4/4$, it is not possible to destabilize the fluid however large the perturbation is!

5.7 Discussion and things to remember

In this Chapter we explored 5 different ways of studying an instability: a local linear stability analysis (which, in this case, did not work), a global linear stability analysis, an energy stability analysis, and finally, two different ways of approaching the question of nonlinear dynamics. These techniques can be applied to any instability study in similar ways, although the details and the outcomes will of course vary.

For instance, there are many instabilities for which a local analysis is indeed very helpful, as in the case of double-diffusive convection, for instance. Meanwhile, there are many systems for which energy stability as introduced here will not give a particularly useful answer – as in the case of stratified shear flows. Finally, weakly nonlinear theory, as we saw, can be quite difficult to investigate – and the case studied here was one of the simpler examples!

If all else fails, of course, it is often possible to study the problem experimentally – either using laboratory experiments, or using numerical experiments.