5.5 Truncated equations and the Lorentz system

As a in alternative path into nonlinear theory, we now explore the idea of using *truncated systems* as a tool to study the time-dependent dynamics of convection close to onset. Indeed, as we saw previously, only very few modes are excited near onset. It may then be possible to model the system dynamics by considering the nonlinear interaction between these few modes only. This procedure will also verify our assumption of the previous section that the steady states found are meaningful representations of the near-onset dynamics.

5.5.1 Derivation of the Lorenz equation

Our previous calculation of the weakly nonlinear *steady-state* just above onset suggests that the modes that matter for small Ra are:

$$\begin{aligned} \phi(x,z,t) &= A(t)\sin(\pi z)\cos(k_c x) \\ T(x,z,t) &= B(t)\sin(\pi z)\sin(k_c x) - C(t)\sin(2\pi z) \end{aligned} \tag{5.119}$$

Plugging these expressions into (6.36), we get

$$\omega = \nabla^2\phi = -(\pi^2+k_c^2)\phi \tag{5.120}$$

and so the momentum equation becomes (recalling that the nonlinear term in the momentum equation is zero)

$$\begin{aligned} -(\pi^2+k_c^2)\dot{A}\sin(\pi z)\cos(k_c x) &= -\mathrm{RaPr}Bk_c\sin(\pi z)\cos(k_c x) \\ &+ \mathrm{Pr}(\pi^2+k_c^2)^2 A\sin(\pi z)\cos(k_c x) \end{aligned} \tag{5.121}$$

which simplifies directly to

$$\dot{A} = \mathrm{RaPr}\frac{k_c}{\pi^2+k_c^2}B - \mathrm{Pr}(\pi^2+k_c^2)A \tag{5.122}$$

The temperature equation on the other hand becomes

$$\begin{aligned} &\dot{B}\sin(\pi z)\sin(k_c x) - \dot{C}\sin(2\pi z) + k_c\pi B\sin(\pi z)\cos(k_c x)A\cos(\pi z)\cos(k_c x) \\ &+ \left[\pi B\cos(\pi z)\sin(k_c x) - 2\pi C\cos(2\pi z)\right]k_c A\sin(\pi z)\sin(k_c x) \\ &-k_c A\sin(\pi z)\sin(k_c x) \\ &= -(\pi^2+k_c^2)B\sin(\pi z)\sin(k_c x) + 4\pi^2 C\sin(2\pi z) \end{aligned} \tag{5.123}$$

Simplifying this, we get:

$$\begin{aligned} &\dot{B}\sin(\pi z)\sin(k_c x) - \dot{C}\sin(2\pi z) + \frac{1}{2}k_c\pi AB\sin(2\pi z) \\ &-\pi k_c AC\sin(k_c x)\left(\sin(3\pi z) - \sin(\pi z)\right) - k_c A\sin(\pi z)\sin(k_c x) \\ &= -(\pi^2+k_c^2)B\sin(\pi z)\sin(k_c x) + 4\pi^2 C\sin(2\pi z) \end{aligned} \tag{5.124}$$

We see that there are 3 types of terms: terms in $\sin(\pi z)$, in $\sin(2\pi z)$ and in $\sin(3\pi z)$. Projecting this equation onto $\sin(\pi z)$ and $\sin(2\pi z)$ respectively (i.e. integrating this equation times $\sin(\pi z)$ or $\sin(2\pi z)$ from $z = 0$ to $z = 1$), and ignoring the term in $\sin(3\pi z)$, we then get

$$\begin{aligned} \dot{A} &= \mathrm{RaPr}\frac{k_c}{\pi^2 + k_c^2}B - \mathrm{Pr}(\pi^2 + k_c^2)A \\ \dot{B} &= k_c A - k_c \pi AC - (\pi^2 + k_c^2)B \\ \dot{C} &= \frac{1}{2}k_c \pi AB - 4\pi^2 C \end{aligned} \tag{5.125}$$

With a little bit of work, it is then possible to rescale these equations into

$$\begin{aligned} a' &= \mathrm{Pr}(-a + b) \\ b' &= ra - b - ac \\ c' &= -sc + ab \end{aligned} \tag{5.126}$$

where $r = \mathrm{Ra}/\mathrm{Ra}_c$, a is proportional to A, b is proportional to rB, c is proportional to rC, and time has been rescaled as well (so the derivative with respect to the new time is denoted by a prime instead of a dot). s is merely a constant that depends on the temporal non-dimensionalization. It is traditionally taken to be 8/3. The new set of equation is very famous: they form *the Lorenz equations.* Note how a represents the amplitude of the convective rolls, b represents the amplitude of the corresponding temperature perturbation, and c corresponds to the change in the horizontally-averaged temperature profile.

It is important to realize that, by contrast with the weakly nonlinear asymptotic expansion of the previous section, there is no *strict or formal* justification for throwing away the $\sin(3z)$ term. We just do it for convenience, in order to close the system of equations. As such, the risk with this approach is that one is never absolutely sure that the terms thrown away do not affect the dynamics of the system at leading order. Truncated systems such as the one derived here should therefore only be seen as toy models. They can be very useful, but have serious limitations as well.

5.5.2 Properties of the Lorenz equations

The Lorenz equations have been studied in depth, and their discovery started the field of *chaotic dynamics.* Let's briefly look into their properties. First, note that they have an obvious fixed point at $a = b = c = 0$ (which corresponds to the state of no convection), which can easily be shown to be stable for $r < 1$ and unstable for $r > 1$. This shows that $r = 1$ marks an important bifurcation in the system. We know this bifurcation, of course – it is the one that corresponds to $\mathrm{Ra} = \mathrm{Ra}_c$.

What happens for $r > 1$? Are there other fixed points? If they do, then the latter must satisfy $a = b$, and $c = ab/s = a^2/s$, and so

$$ra - a - a^3/s = 0 \tag{5.127}$$

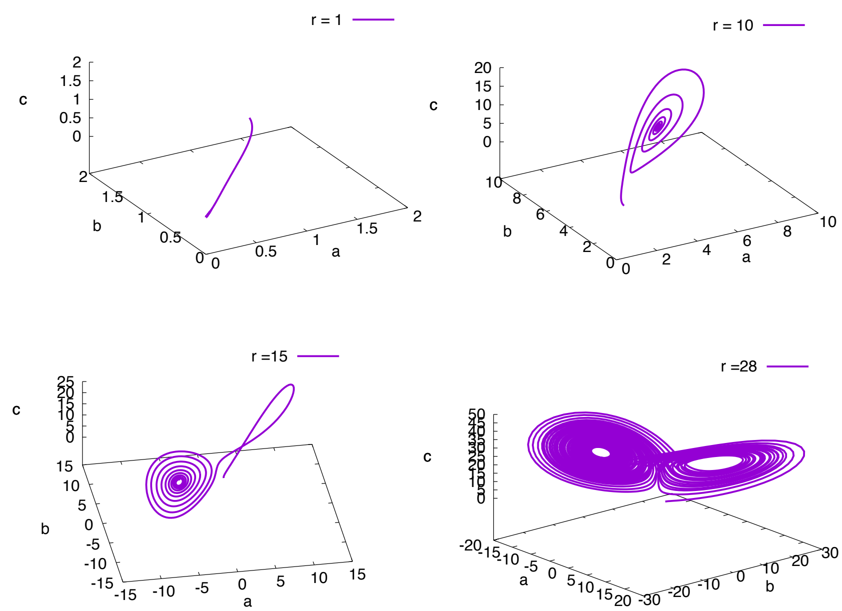

Figure 5.8: Evolution of the Lorenz system for $r = 0.1$, $r = 10$, $r = 15$ and $r = 28$. The first case is subcritical, and the fixed point at the origin is globally stable. The second and third cases, the fixed point at the origin is unstable, but two new fixed points emerge. In the final case all the fixed points are unstable, and the system evolves on a chaotic attractor.

Aside from the $a = 0$ steady-state which we already know of, we see there are two other solutions with $a = \pm s(r-1)$. These solutions only exist when $r \geq 1$, and can be shown to be stable for r between 1 and another bifurcation point (see below). These describe two new *stable* convective states, which correspond to finite amplitude steady convective rolls. This justifies the approach selected in the previous lecture.

Further investigation shows that (at least in the Lorenz system), these new fixed points also become unstable at the critical value

$$r = \frac{\mathrm{Pr}(\mathrm{Pr} + s + 3)}{\mathrm{Pr} - s - 1} \tag{5.128}$$

Beyond that point, *the solutions become chaotic and converge to a strange attractor*. This means that they are strictly *not* periodic, and strictly *not* predictable beyond a certain timescale, but nevertheless have somewhat recognizable patterns. These various dynamical behaviors are illustrated in Figure 5.8, for $\mathrm{Pr} = 10$ and $s = 8/3$. The bifurcation parameter r is varied from 0.1 to 28.