Chapter 5

Instabilities I: Convection

In the following chapters, we will study a few of the fundamental instabilities of fluid dynamics, and learn a number of tools to study their properties. The key questions we will try to answer, each time, are:

- Is the background state considered linearly stable or not? How does this depend on the system parameters?

- Are there finite amplitude instabilities?

- What saturates the instability? What are the properties of the saturated state?

In the vast majority of cases, only the first of these questions can be answered analytically, and even so, not always. However, there are also some interesting instabilities for which all three questions can be answered analytically or semi-analytically, at least approximately.

In this first chapter, we study one of the most famous fluid instabilities, namely convection. Convection is very easily observable in a number of circumstances, including in the kitchen (miso soup, oil starting to warm up, ...), or in the sky (cumulus clouds). It usually occurs when a fluid is heated from below (e.g. the oil), or cooled from the surface (e.g. the miso soup). In this sense, it is clear that convection is an instability that depends crucially on buoyancy.

In what follows, we will study its linear and nonlinear stability, and learn about the properties of turbulent convection.

5.1 Local linear stability analysis of unstably stratified fluids

5.1.1 Introduction to linear instability

In Chapter 4, we studied the propagation of internal gravity waves in stably stratified fluids. We began with the governing Boussinesq equations, namely

$$\begin{aligned}
&\nabla \cdot \boldsymbol{u} = 0 \\
&\rho_m \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \rho \boldsymbol{g} \\
&\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T + w\Theta_{0z} = 0 \\
&\frac{\rho}{\rho_m} = -\alpha T
\end{aligned} \tag{5.1}$$

and assumed that the fluid was stably stratified, so that $\Theta_{0z} > 0$. After linearizing the system, assuming 2D perturbations, and creating a streamfunction (with $\boldsymbol{u} = \nabla \times (\phi \boldsymbol{e}_y)$), we ended up with the following equation describing the evolution of the flow

$$\frac{\partial^2}{\partial t^2}(\nabla^2 \phi) = -N^2 \frac{\partial^2 \phi}{\partial x^2} \tag{5.2}$$

if $N^2 = \alpha g \Theta_{0z}$ is assumed to be constant. As long as this is true, and ignoring the effects of boundary conditions (hence the name *local* analysis) we can then assume that ϕ takes the form

$$\phi(x, z, t) = \hat{\phi}(t) e^{ik_x x + ik_z z} \tag{5.3}$$

which leads to

$$\frac{d^2 \hat{\phi}}{dt^2} = -\frac{N^2 k_x^2}{k^2} \hat{\phi} \tag{5.4}$$

As long as $N^2 > 0$, this equation clearly has oscillatory solutions which have a frequency $\omega = N k_x / k$. These are the gravity waves we studied in Chapter 4.

But what if $N^2 < 0$? While this idea may seems strange at first (why should a square be negative), note that there is no reason why N^2 *should* be positive. It really is the quantity $\alpha g \Theta_{0z}$, and was merely defined to be a square for convenience in the case of internal waves – but there is no reason why Θ_{0z} or α couldn't actually be negative. The first case case could simply correspond to a liquid heated from below, or to a gas for which $dT_0/dz < dT^{\mathrm{ad}}/dz$. The second actually occurs in water below 4° – water close to freezing has very peculiar properties and actually gets *less* dense as the temperature decreases towards 0°.

In all these scenarios, we see that $\hat{\phi}(t)$ could then be an exponential function, instead of an oscillatory function. To be precise, depending on the initial conditions there will be solutions for which $\hat{\phi}(t)$ *increases exponentially with time.* This kind of background is then called *unstable* to perturbations. Seeking solutions of the kind $\hat{\phi}(t) = \hat{\phi}_0 \exp(\lambda t)$, we get

$$\lambda^2 = -\alpha g \Theta_{0z} \frac{k_x^2}{k_x^2 + k_z^2} \tag{5.5}$$

so

$$\lambda = \pm\sqrt{-N^2}\frac{k_x}{k} \tag{5.6}$$

The (positive) value of λ thus derived is called the *growth rate* of the instability. We immediately see that for every positive solution for λ, there is also a negative one. But we are of course much more interested in the positive λ case, since its existence tells us that there are perturbations that can grow.

5.1.2 A physical explanation for the instability

The basic local convective instability is very easy to understand, and is illustrated in Figure 5.1. If a parcel of fluid from the lower, hotter regions, is moved up a little bit, it finds itself in a cooler environment, and is therefore less dense than its surroundings. The buoyancy force pushes it upwards, and the process repeats in a positive feedback loop. The parcel continues to move upwards and accelerates.

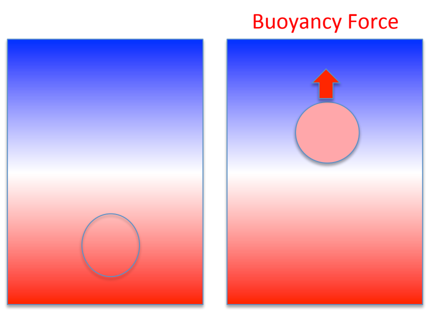

Figure 5.1: A schematic view of local convection

5.1.3 The fastest growing modes

The relationship between the growth rate λ and mode structure (given by $\boldsymbol{k}$) obtained above is sometimes also called a dispersion relation even in the context of instabilities, simply by analogy with the equivalent relationship for waves. It give us a lot of information about what to expect from the early development of a fluid instability.

Indeed, one can always expect that a fluid system is full of very low amplitude noise. Hence, the initial conditions for an instability always contain all sorts of possible modes that either grow or decay. In the case of convection, we see that there is no restriction on the values of $\boldsymbol{k}$ for which the mode could grow – for any initial $\boldsymbol{k}$, there is a corresponding positive value of λ. This is actually fairly unphysical, as we shall see shortly. Nevertheless – in this simple model, all modes grow as long as $N^2 < 0$. However, they do not all grow *equally* – there are modes that grow more rapidly than others. These are called the *fastest-growing modes*.

Fastest-growing modes are of interest because, although the initial noise may contain a spectrum of modes, all with roughly the same amplitude, the exponential amplification will be much larger for rapidly-growing modes than for slowly growing ones. Hence, after some time, the system will be dominated by a few modes only, those with the largest growth rate.

In this local convection model, we see that the growth rate λ does not depend on the mode wavenumber, but rather, on the orientation of $\boldsymbol{k}$. This result is somewhat artificial – had we taken diffusion and viscosity into account, we would have found that the growth rate was lower for small-scale modes than for large-scale modes (see Homework). Nevertheless, let's proceed with λ as derived. It's easy to show that, in this case, λ is maximum for $k_z = 0$, and takes the value

$$\lambda_{\rm fgm} = \sqrt{-N^2} \tag{5.7}$$

The fastest-growing convective modes appear to have no structure in the vertical direction! It therefore looks quite different from our simplistic parcel argument. In reality the entire fluid column has the same temperature, and the same velocity. Furthermore, because $\nabla \cdot \boldsymbol{u} = 0$, if $\partial w/\partial z = 0$, then $\partial u/\partial x = 0$. Unless $k_x = 0$ this then implies that $u = 0$ – in other words, fluid is only allowed to move up and down, not laterally. Because of this peculiar structure, these linearly unstable fastest-growing local modes of convection are commonly-called *elevator modes*.

Already, we should see that there is something rather fishy about them. We began by assuming that we were looking at a local instability (ie. an instability that we can model by ignoring the boundary conditions), but the modes themselves are vertically invariant – which is going to be inconsistent as soon as we try to apply boundary conditions in the z direction. More on this later.

5.1.4 Nonlinear saturation

Of course, in reality we do not expect the perturbations to grow exponentially forever – this would violate a few important laws of Physics. The reason we were able to get exponential solutions here lies in the fact that we have linearized the original governing equations. As the perturbations grow, they will gradually become large enough for the neglected nonlinear terms to become significant, at which point the linearized equations will no longer be valid, and the exponential growth should stop.

A simple example of nonlinear saturation

To see how this may work, let's study a very simply 1D ODE. Consider the equation

$$\frac{df}{dt} = af - bf^2 \tag{5.8}$$

If we linearize this ODE (for small enough f) we find that it can be approximated by $df/dt = af$, which has a growing exponential solution provided $a > 0$. Let's assume that's the case.

Even if the initial value of f is very small, say, $f(0) = \epsilon$, after a while, $f(t) = \epsilon e^{at}$ will become quite large. Large enough, in fact, for the neglected nonlinear terms to become important. This happens when

$$af \simeq bf^2 \to \epsilon e^{at} \simeq a/b \to t \simeq (1/a)\ln(a/\epsilon b) \tag{5.9}$$

At that point, f stops growing exponentially, and the instability saturates.

In this particular example, saturation can be calculated analytically. Indeed, this equation has the solution

$$\frac{df}{af - bf^2} = dt \to f(t) = \frac{\epsilon e^{at}}{b\epsilon e^{at}/a + 1} \tag{5.10}$$

We see that for small t, $f(t) \simeq \epsilon e^{at}$ as expected, but for large t,

$$f(t) \to \frac{a}{b} \tag{5.11}$$

In other words, $f(t)$ stops growing exponentially and instead converges to another fixed point, non-zero this time, that arise directly from the balance between the linear terms (which drive the instability) and the nonlinear terms (which act to damp it, in this example). The function $f(t)$, for $a = b = 1$ and $\epsilon = 0.01$ is shown in Figure 5.2

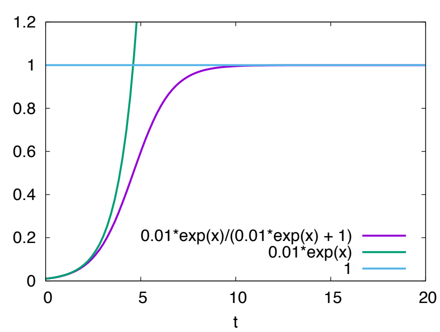

Figure 5.2: The solution of $f' = f - f^2$ with $f(0) = 0.01$. At early times, the function $f(t)$ grows exponentially as $f(t) \simeq 0.01e^t$, but at late times the growth saturates to $f(t) = 1$.

Nonlinear saturation of the elevator modes?

Let's see if, in our example, the neglected nonlinear terms could be responsible for the saturation of the instability. To do this, we need to go back to our original set of equations, and determine when quantities such as $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\boldsymbol{u} \cdot \nabla T$ become comparable to other such as $w\Theta_{0z}$ for instance. A good way of doing this is to plug in the linear solutions for T and $\boldsymbol{u}$ into these expressions.

If $\phi = \phi_0 e^{ik_x x + \lambda t}$ (recalling that elevator modes have $k_z = 0$) then

$$u = 0 \text{ and } w = -\frac{\partial \phi}{\partial x} = -ik_x \phi_0 e^{ik_x x + \lambda t} \tag{5.12}$$

Hence

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = w \frac{\partial}{\partial z}(-ik_x \phi_0 e^{ik_x x + \lambda t} \boldsymbol{e}_z) = 0 \tag{5.13}$$

so the nonlinear term in the momentum equation is identically zero for the elevator mode! To estimate $\boldsymbol{u} \cdot \nabla T$, we apply a similar method. Since $\boldsymbol{u} = w\boldsymbol{e}_z$, then $\boldsymbol{u} \cdot \nabla T = w\partial T/\partial z$, but the elevator modes are invariant in z, so this term is 0 too!

This suggests that the nonlinear terms in this system are identically zero for the elevator modes, and the latter can therefore *never saturate on their own*! Another way of putting it is to note that elevator modes are exact nonlinear solutions of the governing equations (and not just of the linear equations). This, combined with the fact that they can't easily be fitted to boundary conditions, makes them very fishy indeed.

As it turns out, while being nice and simple to study, our model this time was *oversimplified*. By removing all effects of the boundary conditions, we ended up neglecting important physics, and the results became *unphysical*. This is not always the case – some instabilities *can* be studied using local analysis, and yield meaningful results, but convection isn't one of them; that's because the basic convective perturbation has a tendency to spread out, and would span the entire space if it could. As a result, it *has* to know about the boundary conditions applied to the system.

Having established this, we now go back a few steps and put back some of the important neglected physics (such as boundary conditions and diffusion) into the problem. This actually recovers the original description of the convective instability, first studied by Rayleigh.

5.2 Rayleigh-Bénard convection

This section is adapted from the textbook Introduction to Hydrodynamic Stability by Drazin.

5.2.1 Model setup

We now consider the more physically-realistic problem of convection in a liquid between two horizontal plane parallel plates, separated by a distance H. The bottom plate is held at a larger temperature than the top one, hence driving convection (assuming $\alpha > 0$). Both plates are impermeable. Horizontally, we could assume a number of boundary conditions, but for simplicity, we assume the horizontal domain is infinite so that the horizontal mode structure is of the kind $\exp(ik_x x)$.

The complete set of equations governing fluid motion and temperature fluctuations about the mean T_m, in this system, are the Boussinesq equations for liquids,

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0 \\ &\rho_m \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \rho \boldsymbol{g} + \rho_m \nu \nabla^2 \boldsymbol{u} \\ &\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa_T \nabla^2 T \\ &\frac{\rho}{\rho_m} = -\alpha T \end{aligned} \tag{5.14}$$

where this time we keep the viscous and thermal diffusion terms. The boundary conditions in the vertical direction (so far) are $w = 0$ on both plates, $T = T_m + \Delta T/2$ on the lower plate (at $z = 0$), and $T = T_m - \Delta T/2$ on the top plate (at $z = H$). A schematic of the model is shown in Figure 5.3.

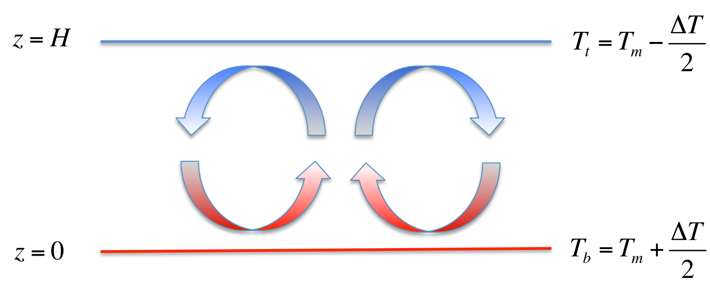

Figure 5.3: Schematic model of Rayleigh-Benard convection.

The background steady state around which we will be perturbing is one in which there is no fluid motion, hence $\bar{\boldsymbol{u}} = 0$. In that case, temperature is forced to diffuse from one boundary to the next, satisfying

$$\kappa_T \frac{d^2 \bar{T}}{dz^2} = 0 \tag{5.15}$$

The solution to this equation is a linear temperature profile. To satisfy the boundary conditions, we must have $\bar{T}(z) = -\Delta T z/H + T_m + \Delta T/2$. Plugging this solution into the equation of state, we see that it corresponds to a background density profile

$$\bar{\rho}(z) = \alpha \rho_m \left(\Delta T z/H - T_m - \Delta T/2 \right) \tag{5.16}$$

If ΔT and α are both positive (which we assume here), we see that $\bar{\rho}$ increases with height – in other words, fluid near the top plate is heavier than fluid near the bottom plate. As we shall see, this is crucial for the instability to develop. Finally, note that a background density gradient implies that there must also be a background pressure gradient, satisfying hydrostatic equilibrium:

$$\frac{d\bar{p}}{dz} = -\bar{\rho}g \tag{5.17}$$

We do not actually have to solve for $\bar{p}$, but merely acknowledge that it exists, and remember that this equation is satisfied.

5.2.2 Linear perturbations

We now perturb this background, assuming $\boldsymbol{u} = \tilde{\boldsymbol{u}}$ and $T = \tilde{T} + \bar{T}$. The boundary conditions on the perturbations are the original ones minus those already satisfied by the background state, hence $\tilde{u} = \tilde{w} = 0$ on both plates, and $\tilde{T} = 0$ on both plates. After the usual work, the linearized equations become

$$\begin{aligned}
&\nabla \cdot \tilde{\boldsymbol{u}} = 0 \\
&\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} = -\nabla \tilde{p} + \alpha \tilde{T} g \boldsymbol{e}_z + \nu \nabla^2 \tilde{\boldsymbol{u}} \\
&\frac{\partial \tilde{T}}{\partial t} + w\frac{d\bar{T}}{dz} = \kappa_T \nabla^2 \tilde{T}
\end{aligned} \tag{5.18}$$

As before, we assume the flow is 2D, and use a streamfunction to describe $\tilde{\boldsymbol{u}}$. Since the background is invariant in time, we now directly assume that all dependent variables are also proportional to $\exp(\lambda t)$, hence we have

$$\tilde{u} = \frac{d\hat{\phi}}{dz} e^{ik_x x + \lambda t} \ , \ \tilde{w} = -ik_x \hat{\phi} e^{ik_x x + \lambda t} \text{ and } \tilde{q} = \hat{q} e^{ik_x x + \lambda t} \tag{5.19}$$

where q is any of the three thermodynamic variables. Plugging this into the governing linearized equations, we have

$$\begin{aligned}
&\lambda \frac{d\hat{\phi}}{dz} = -\frac{ik_x}{\rho_m}\hat{p} + \nu \frac{d}{dz}\left(\frac{d^2\hat{\phi}}{dz^2} - k_x^2 \hat{\phi}\right) \\
&-ik_x \lambda \hat{\phi} = -\frac{1}{\rho_m}\frac{d\hat{p}}{dz} + \alpha g \hat{T} - ik_x \nu \left(\frac{d^2\hat{\phi}}{dz^2} - k_x^2 \hat{\phi}\right) \\
&\lambda \hat{T} - ik_x \hat{\phi} \Theta_{0z} = \kappa_T \left(\frac{d^2 \hat{T}}{dz^2} - k_x^2 \hat{T}\right)
\end{aligned} \tag{5.20}$$

Eliminating $\hat{p}$ between these two equations yields

$$\begin{aligned}
&\lambda D\hat{\phi} = -i\alpha g \hat{T} k_x + \nu D^2 \hat{\phi} \\
&-ik_x \hat{\phi} \Theta_{0z} = \kappa_T D\hat{T} - \lambda \hat{T}
\end{aligned} \tag{5.21}$$

where we have defined the operator $D = d^2/dz^2 - k_x^2$. Finally, we can eliminate, say, $\hat{T}$, to get

$$(\lambda - \kappa_T D)(\lambda - \nu D)D\hat{\phi} = N^2 k_x^2 \hat{\phi} \tag{5.22}$$

where we have used the standard substitution $N^2 = \alpha g d\bar{T}/dz = -\alpha g \Delta T/H$. With the assumptions made here, N^2 is negative

We therefore see that, in order to find the growth rates of the convective modes of instability (each of which is characterized by its horizontal wavenumber k_x), we have to solve a 6th order ODE in z, for the eigenvalue λ. Solving a 6th-order ODE requires 6 boundary conditions – however, here we only have 4 so far (two on $\tilde{T}$ and two on $\tilde{w}$), neither of which are expressed in terms of $\hat{\phi}$ yet! So we have a bit more work to do before we can solve for λ.

First note that $\tilde{w} = 0$ is equivalent to $-ik_x\hat{\phi} = 0$. For any $k_x \neq 0$, this requires $\hat{\phi} = 0$. Plugging $\hat{T} = 0$ and $\hat{\phi} = 0$ in the thermal energy equation shows that these conditions imply that $D^2\hat{\phi} = 0$, which is itself equivalent to requiring that $d^2\hat{\phi}/dz^2 = 0$ at the plates. For the final two boundary conditions, we have two options. These conditions will have to be applied to the horizontal velocity near the plate, and can either be *no slip* (meaning that the fluid velocity very near the plate must be equal to the velocity of the plate – here, this implies $\tilde{u} = 0$ at the plate), or *stress-free* (meaning that there is no viscous stress communicated from the plate to the fluid – here, this implies that $d\tilde{u}/dz = 0$ at the plate).

No-slip boundary conditions are usually the more physically realistic ones (unless modeling the interface between two fluids, which we are not doing here). However, in this particular case, we will use stress-free boundary conditions because they yield very simple analytical solutions to the problem (while no-slip ones require somewhat more elaborate calculations). Indeed, the advantage of the stress-free condition $d\tilde{u}/dz = 0$ is that they end up being equivalent to the temperature boundary condition $d^2\hat{\phi}/dz^2 = 0$, since $\tilde{u} \propto d\hat{\phi}/dz$. While this choice affects the quantitative results we are about to obtain, it does not affect them qualitatively (cf Homework).

It's reasonably easy to see that, with these boundary conditions, the vertical eigenmodes are simply

$$\hat{\phi}_n(z) = \sin\left(\frac{n\pi z}{H}\right) \tag{5.23}$$

and that λ_n (the growth rate of the n-th mode) is the solution of

$$(\lambda_n + \kappa_T k^2)(\lambda_n + \nu k^2)k^2 = -N^2 k_x^2$$
$$\rightarrow \lambda_n^2 + \lambda_n k^2(\nu + \kappa_T) + \nu\kappa_T k^4 + N^2\frac{k_x^2}{k^2} = 0 \tag{5.24}$$

where

$$k^2 = k_x^2 + \frac{n^2\pi^2}{H^2} \tag{5.25}$$

This equation is a quadratic for λ_n, which has the solutions

$$\lambda_n = -\frac{1}{2}k^2(\nu + \kappa_T) \pm \sqrt{\frac{1}{4}k^4(\nu - \kappa_T)^2 - N^2\frac{k_x^2}{k^2}} \tag{5.26}$$

This expression reveals a number of important results about the convective modes. First, since we have assumed that $N^2 < 0$, λ_n is always real. This means that the modes will either grow or decay exponentially. Secondly, while decaying modes always exist, growing modes do not. Additional conditions must be satisfied for instability to take place.

5.2.3 Onset of convective instability (linear regime)

A necessary condition for a positive λ_n to exist is:

$$\frac{1}{4}k^4(\nu+\kappa_T)^2 < \frac{1}{4}k^4(\nu-\kappa_T)^2 - N^2\frac{k_x^2}{k^2} \tag{5.27}$$

which can be re-written as

$$\alpha g \frac{\Delta T}{H} = -N^2 > \frac{(k_x^2+n^2\pi^2/H^2)^3}{k_x^2}\nu\kappa_T \tag{5.28}$$

This expression implies that, *for fixed* k_x, a larger ΔT is needed to destabilize a mode with large n than one with small n. In fact, the first mode to be destabilized is the fundamental, which has $n = 1$. Furthermore, if we *fix* $n = 1$, then we see that the minimum value of ΔT needed to destabilize the system is given by

$$\frac{\alpha g \frac{\Delta T_{\rm min}}{H}}{\nu\kappa_T} = \frac{(k_x^2+\pi^2/H^2)^3}{k_x^2} \tag{5.29}$$

To see things a bit more clearly, let's re-define the normalized horizontal wavenumber as $a = Hk_x$. Then

$$\frac{\alpha g \Delta T_{\rm min} H^3}{\nu\kappa_T} = \frac{(a^2+\pi^2)^3}{a^2} \tag{5.30}$$

The quantity on the right-hand-side is non-dimensional, which implies that the one on the left must also be non-dimensional. As it turns out,

$$\mathrm{Ra} = \frac{\alpha g \Delta T H^3}{\nu\kappa_T} \tag{5.31}$$

is one of the most important non-dimensional numbers in fluid dynamics: the *Rayleigh number* (named after Lord Rayleigh, who first discussed it). We see that it only depends on input system parameters, not on the properties of the emerging instabilities. However, because it *controls* whether the system is stable or not (see below), it is one of the *control parameters* or *bifurcation parameters* for the system.

Coming back to the question of the necessary condition for instability, note that for a mode with nondimensional wavenumber a to be unstable, we need to have

$$\mathrm{Ra} > \mathrm{Ra}_{\rm min}(a) = \frac{(a^2+\pi^2)^3}{a^2} \tag{5.32}$$

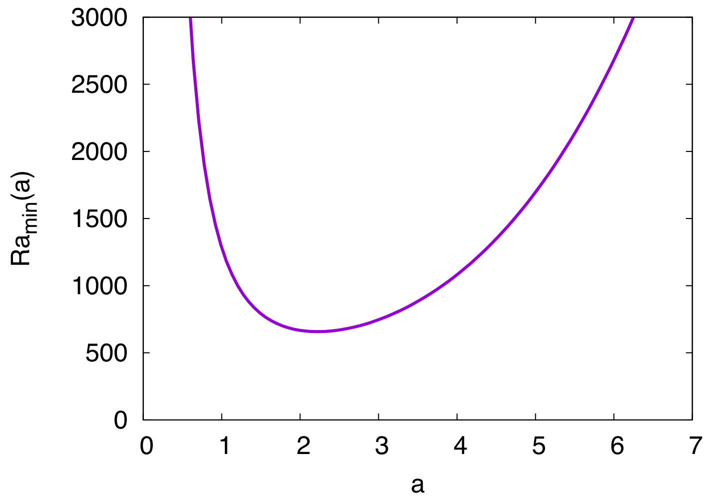

Figure 5.4: Critical Rayleigh number for convective instability as a function of the mode wavenumber $a = Hk_x$.

The curve $\mathrm{Ra}_{\mathrm{min}}(a)$ is shown in Figure 5.4. We see that it has an absolute minimum at some particular value a_c. This means that the modes of horizontal wavenumbers a_c/H are the ones that are most easily destabilized (since they require the smallest value of ΔT all other parameters being fixed, for instance).

Minimizing $\mathrm{Ra}_{\mathrm{min}}(a)$ over all possible values of a gives us the absolute criterion for linear instability as $\mathrm{Ra} > \mathrm{Ra}_c$, where

$$\mathrm{Ra}_c = \min_{a^2} \frac{(a^2+\pi^2)^3}{a^2} = \frac{27\pi^4}{4} \tag{5.33}$$

This value of Ra_c is achieved for $a_c = \pi/\sqrt{2} \simeq 2.22$. Had we assumed somewhat different boundary conditions (no-slip, for instance), we would have obtained a different value for Ra_c and a_c. Since the stress-free ones assume that the fluid is free to slip against the boundary without any friction, they are the least restrictive ones, and lead to the lowest Ra_c. For no-slip conditions, $\mathrm{Ra}_c \simeq 1700$ instead, and $a_c \simeq 3.1$ (cf. Homework).

The Rayleigh number

Why is the Rayleigh number the relevant bifurcation parameter for convection? The answer lies in understanding the balance of forces present in the momentum equation. In addition to the usual pressure gradient, we see that two forces are present: the buoyancy force, which drives the instability, and the viscous force, which attempts to suppress it. As it turns out, the Rayleigh number controls the relative amplitude of the two.

To see why, note that near the onset of convection, nonlinear terms *and* time derivatives are negligible (because λ is close to 0). Hence we have

$$\begin{aligned}
&\nabla \cdot \tilde{\boldsymbol{u}} = 0 \\
&\nabla \tilde{p} \simeq \alpha \tilde{T} g \boldsymbol{e}_z + \nu \nabla^2 \tilde{\boldsymbol{u}} \\
&w \frac{d\bar{T}}{dz} \simeq \kappa_T \nabla^2 \tilde{T}
\end{aligned} \tag{5.34}$$

From a dimensional analysis, assuming that typical vertical and horizontal lengthscales are the same and close to H, we get from the thermal energy equation that

$$w \frac{d\bar{T}}{dz} \sim \kappa_T \frac{\tilde{T}}{H^2} \tag{5.35}$$

which implies that

$$\tilde{T} \sim H^2 \frac{\Delta T}{H} \frac{w}{\kappa_T} \tag{5.36}$$

We then see that the ratio of the buoyancy term to the viscous term, in the momentum equation, is dimensionally of the order of

$$\frac{|\alpha g \tilde{T} \boldsymbol{e}_z|}{|\nu \nabla^2 \boldsymbol{u}|} \sim \frac{\alpha g H^2 \frac{\Delta T}{H} \frac{w}{\kappa_T}}{\nu \frac{|u|}{H^2}} \sim \mathrm{Ra} \tag{5.37}$$

as long as $|u| \sim w$. Because of incompressibility, this is satisfied whenever the vertical and horizontal lengthscales are of the same order (which was assumed earlier)

A small Ra corresponds to fluid systems where the buoyancy force is much smaller than the viscous force. It is therefore not surprising to note that this limit also corresponds to the case that is stable to convection. On the other hand when Ra is large, the buoyancy force dominates, and the convective instability can occur.

5.2.4 Fastest-growing modes

Beyond the onset of instability, that is, for a fixed value of $\mathrm{Ra} > \mathrm{Ra}_c$, the fastest growing modes can be found by maximizing λ over all possible k_x and $k_z = n\pi/H$. To do so, first note that λ_n can be rewritten as

$$\lambda_n \frac{H^2}{\kappa_T} = -\frac{a^2 + n^2\pi^2}{2} \left(\frac{\nu}{\kappa_T} + 1 \right) + \frac{1}{2} \sqrt{(a^2 + n^2\pi^2)^2 \left(\frac{\nu}{\kappa_T} - 1 \right)^2 - \frac{N^2 H^4}{\kappa_T^2} \frac{4a^2}{a^2 + n^2\pi^2}} \tag{5.38}$$

We have multiplied λ_n by H^2/κ_T, so that the LHS is now non-dimensional (the unit time being the thermal diffusion timescale across H). The RHS reveals another well-known number called the *Prandtl* number (after Prandtl):

$$\mathrm{Pr} = \frac{\nu}{\kappa_T} \tag{5.39}$$

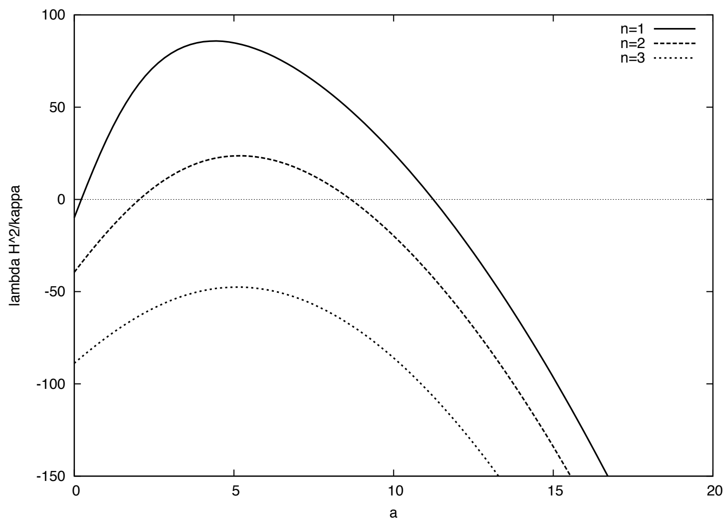

Figure 5.5: Non-dimensional growth rate $\lambda H^2/\kappa_T$ for $n = 1$, 2 and 3, for Ra = 2000 and Pr = 1. This kind of diagram representing growth rate vs. wavenumber is fairly typical of most instabilities.

in which case

$$\lambda_n \frac{H^2}{\kappa_T} = -\frac{a^2+n^2\pi^2}{2}(\mathrm{Pr}+1) + \frac{1}{2}\sqrt{(a^2+n^2\pi^2)^2(\mathrm{Pr}-1)^2 + \mathrm{Pr}\mathrm{Ra}\frac{4a^2}{a^2+n^2\pi^2}} \tag{5.40}$$

The figure below show how λ_n varies with k_x for different values of n, for Pr = 1 and Ra = 2000. We see that, at these parameter values, only $n = 1$ and $n = 2$ are destabilized, and that, for a given k_x, λ is always[1] largest for $n = 1$. This means that we can find the fastest-growing modes by setting $n = 1$, and maximize λ with respect to k_x.

To do so, it's easier to use the original quadratic (5.24). Taking its derivative with respect to k_x^2, and setting $d\lambda/d(k_x^2) = 0$, we get

$$\lambda_1(\nu+\kappa_T) + 2\nu\kappa_T k^2 + \frac{N^2\pi^2}{H^2k^4} = 0 \tag{5.41}$$

This equation, together with (5.24) with $n = 1$, forms a system of two equations for two unknown variables, the growth rate λ and wavenumber k_x of the fastest-

[1]Of course, we have not proved that for all values of Ra and Pr. This can also be done formally, but it's a little bit harder.

growing modes. Unfortunately they cannot easily be solved analytically, but can be solved numerically using, for instance, a Newton Method (cf. Homework).

The technique described here is a fairly standard method for finding the most rapidly-growing modes of linear instabilities, when λ has a well-defined maximum for a finite $\boldsymbol{k}$. This is not always the case, however. We will see that in some cases, λ can be maximum for $k = 0$ (an *infrared catastrophe*) or for $k \rightarrow \infty$ (an *ultraviolet catastrophe*). Usually, however, systems which have such a catastrophe are ill-posed, in the sense that some physics have been neglected that would otherwise prevent it. An infrared catastrophe is usually impossible in any finite-size system (since the domain size itself sets the smallest possible value of k for the instability). An ultraviolet catastrophe is usually prevented by taking into account dissipation (which preferentially damps small-scales, and prevent them from growing). This was illustrated here quite well: the case of the local convective instability turned-out to have a number of problems, which are all solved simply by taking into account more relevant physics (domain boundaries and dissipation).

The bottom line is that it is always preferable to include more physics, but this can, of course, make the problem too hard to solve analytically. This is a good example of how the art in applied mathematics lies in knowing how to find the right trade-off between physical realism and mathematical tractability.