4.2 Surface waves

Surface waves are probably the most common example of waves we are visually exposed to in everyday life – from small-scale waves in the bathtub to medium-scale waves approaching a the beach, to very large-scale waves travelling on the surface of the deep ocean. However, because they propagate on the interface between a liquid and air, their mathematical description is significantly more complicated than that of pressure or internal gravity waves, which is why we have deferred studying them until now.

4.2.1 Derivation of the wave equation for surface waves

This section is adapted from the Geophysical Fluid Dynamics Summer Program 2009 Lectures (specifically, Lecture 1), given by Harvey Segur, and written up by Michael Bates.

We begin our derivation by considering a layer of liquid. It is located above a bottom boundary, whose equation is $z = -h(x, y)$. At rest, the surface of the liquid is at $z = 0$. When waves are present, on the other hand, the surface undulates, and its equation is given by $z = \eta(x, y, t)$. The liquid is incompressible, so that $\nabla \cdot \boldsymbol{u} = 0$ within it. We will also neglect all density perturbations entirely, so that ρ is constant. The setup is illustrated in Figure 4.1.

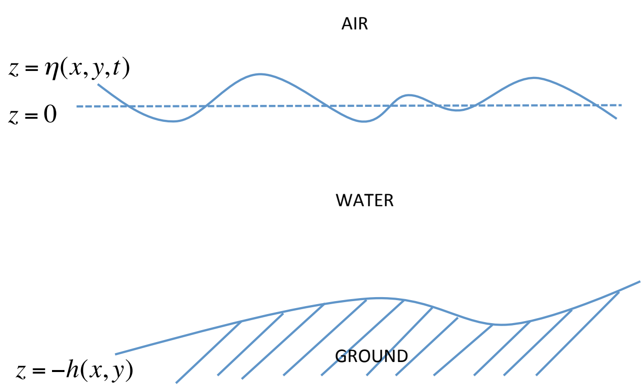

Figure 4.1: Model setup

For simplicity, we will also assume that at time $t = 0$, any fluid motion in the liquid is irrotational, which means that its vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 0$. This restricts the class of initial conditions, and therefore the class of waves we are considering, but will significantly simplify the mathematical description of the model. Indeed, the equation of motion in the limit where viscosity is negligible, is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \boldsymbol{g} \tag{4.1}$$

Since ρ is constant, taking the curl of this equation we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \tag{4.2}$$

This shows that if, at $t = 0$ $\omega = 0$ then ω remains zero at all times afterwards. In other words, in an incompressible non-rotating and non-viscous fluid it is not possible to create any vorticity if none is originally present.

The advantage of considering irrotational flows is that they can be written as

$$\boldsymbol{u} = \nabla \phi \tag{4.3}$$

where ϕ is any scalar function of (x, y, z, t). Furthermore, incompressibility then implies $\nabla^2 \phi = 0$. Plugging this into the momentum equation, and writing each term as a gradient (using the fact that ρ is constant) we get

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + gz \right) = 0. \tag{4.4}$$

Integrating this expression with respect to all spatial variables gives

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + gz = F(t) \tag{4.5}$$

where $F(t)$ is an integration function of time only. Note that since the velocity potential ϕ is defined up to an arbitrary additive function of time, we may absorb $F(t)$ into $\partial \phi / \partial t$. We thus recover the well-known *Bernoulli's Law.*

We assume that the bottom boundary is impermeable, and thus, enforce a "no-normal flow" boundary condition,

$$\boldsymbol{u} \cdot \nabla \left(z + h(x, y) \right) = 0 \quad \text{at } z = -h(x, y) \,, \tag{4.6}$$

where $\nabla \left(z + h(x, y) \right)$ is the normal vector to the bottom surface. Using the definition of the velocity potential, equation (4.3), we obtain,

$$\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial h}{\partial y} = 0 \rightarrow \frac{\partial \phi}{\partial z} + \nabla \phi \cdot \nabla h = 0 \,. \tag{4.7}$$

On the surface, $z = \eta(x, y, t)$, we require the continuity of the pressure field p. Just above the surface, there are two contribution to pressure: a pressure due to the weight of the atmosphere and a pressure given by the surface tension, which conceptually acts like an elastic membrane stretched over the surface of the liquid.

$$p = p_{\text{air}} - \sigma \nabla \cdot \hat{\boldsymbol{n}} \quad \text{at } z = \eta(x, y, t) \,, \tag{4.8}$$

where the second term is the surface tension Young-Laplace term, and σ is a constant representing its strength, with units Nm^{-1}, and $\hat{\boldsymbol{n}}$ is the surface normal unit vector. For water at room temperature, $\sigma \simeq 70$ N/m. From here on, we assume that $p_{\rm air} = 0$, again without loss of generality. Note that we are ignoring the effects of wind. Vector calculus tells us that

$$\hat{\boldsymbol{n}} = \frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\,. \tag{4.9}$$

Using (4.8) in Bernoulli's Law (4.5) gives us the dynamic boundary condition on the free surface,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{\sigma}{\rho}\nabla\cdot\left\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right\}. \tag{4.10}$$

Finally, we obtain a kinematic boundary condition by assuming that a material element on the free surface stays on the free surface,

$$\frac{D}{Dt}\left(z(t) - \eta(x,y,t)\right) = 0. \tag{4.11}$$

Noting that since $Dz/Dt = w = \partial_z\phi$ and $D\eta/Dt = \partial_t\eta + \boldsymbol{u}\cdot\nabla\eta$, we obtain

$$\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta = \frac{\partial\phi}{\partial z}\,, \tag{4.12}$$

when evaluated at $z = \eta$.

To summarize, the governing equations are

$$\begin{aligned}
\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta &= \frac{\sigma}{\rho}\nabla\cdot\left\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right\} \quad \text{on} \quad z = \eta(x,y,t) \\
\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta &= \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = \eta(x,y,t) \\
\nabla^2\phi &= 0 \text{ for } -h(x,y) < z < \eta(x,y,t) \\
\frac{\partial\phi}{\partial z} + \nabla\phi\cdot\nabla h &= 0 \text{ on } \quad z = -h(x,y).
\end{aligned}$$

The first equation relates the evolution of the velocity potential for a material element on the surface to the restoring force of gravity and the surface tension. The second equation describes the kinematic evolution of the free surface. The third equation is the continuity equation, where we have assumed that the fluid is incompressible and irrotational. The last equation is a statement that we do not allow any flow across the impermeable bottom boundary. This equations are highly nonlinear, but could in principle describe the evolution of high-amplitude waves, as long as their surface is a single-valued function (this rules out overturning waves). In order to make progress in studying these waves, however, we first linearize them and only consider small-amplitude perturbations.

4.2.2 Small-amplitude surface waves in a layer of constant depth.

This section is adapted from the Geophysical Fluid Dynamics Summer Program 2009 Lectures (specifically, Lecture 2), given by Harvey Segur and written up by Michael Bate and Ali Mashayekhi.

In this section, we will assume for simplicity that h is constant. The bottom boundary condition then becomes $\partial\phi/\partial z = 0$ at $z = -h$. The results can be generalized using the wave-packet approximation if h is a slowly varying function of position. This will be one of the projects for the course.

Plane wave solutions

Assuming that all perturbations are small amplitude, we can linearize the governing equations by neglecting any nonlinear term in η or ϕ. We get

$$\begin{aligned}
\frac{\partial\phi}{\partial t} + g\eta &= \frac{\sigma}{\rho}\nabla^2\eta \text{ on } \quad z = \eta(x,y,t) \\
\frac{\partial\eta}{\partial t} &= \frac{\partial\phi}{\partial z} \text{ on } \quad z = \eta(x,y,t) \\
\nabla^2\phi &= 0 \text{ for } -h(x,y) < z < \eta(x,y,t) \\
\frac{\partial\phi}{\partial z} + \nabla\phi\cdot\nabla h &= 0 \text{ on } \quad z = -h(x,y).
\end{aligned}$$

Now, applying "boundary conditions" at the free surface is a little bit awkward, since η itself is an unknown of the problem. However, if the waves are small amplitude, then note that by Taylor expansion

$$\phi(x,y,\eta,t) = \phi(x,y,0,t) + \eta(x,y,t)\left.\frac{\partial\phi}{\partial z}\right|_{z=0} + ... \tag{4.13}$$

so that the first and second equations, in the linear approximation, can actually be evaluated at $z = 0$. We can now combine these two equations into a single one for ϕ for instance, as

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\phi}{\partial z} + \frac{\sigma}{\rho}\nabla^2\frac{\partial\phi}{\partial z} \text{ on } \quad z = 0 \tag{4.14}$$

We see from this expression that we can expect two limits, one where gravity (first term on the RHS) dominates and the other where surface tension (second term on the RHS) dominates. The former are called *surface gravity waves* and the latter *capillary waves*.

The linear surface wave problem with constant depth is effectively a problem in which one should solve Laplace's equation in a plane-parallel domain, subject to time- and spatially-dependent boundary conditions. To do so, note that by separation of variables, and since the background is invariant in x, y and t (as long as σ and h are constant), we must have

$$\phi(x,y,z,t) = \int dk_x \int dk_y \hat{\phi}(z;\boldsymbol{k}) e^{ik_x x + ik_y y - i\omega t} \tag{4.15}$$

where $\boldsymbol{k} = (k_x, k_y)$, and $\hat{\phi}(z;\boldsymbol{k})$ satisfies

$$\frac{\partial^2 \hat{\phi}}{\partial z^2} - k^2 \hat{\phi} = 0 \text{ for } -h < z < 0 \tag{4.16}$$

where $k = |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2}$, and $\hat{\phi}$ is subject to boundary conditions

$$-\omega^2 \hat{\phi} = -g \frac{d\hat{\phi}}{dz} + \frac{\sigma}{\rho} \nabla^2 \frac{d\hat{\phi}}{dz} \text{ on } \quad z = 0 \tag{4.17}$$

and

$$\frac{d\hat{\phi}}{dz} = 0 \text{ on } \quad z = -h \tag{4.18}$$

The equations for $\hat{\phi}$, together with the lower boundary condition gives

$$\hat{\phi}(z;\boldsymbol{k}) = A(\boldsymbol{k}) \cosh\left(k(z+h)\right) \tag{4.19}$$

where A is the amplitude of the mode with wavenumber $\boldsymbol{k}$. For large hk (that is, for deep water), then $\cosh(k(z+h)) \simeq e^{k(z+h)}$ near the surface (near $z = 0$), so $\cosh(k(z+h)) \simeq e^{k(z+h)} = e^{kh} e^{kz}$. This shows that all the components of the wave velocity decay more-or-less exponentially on a lengthscale $1/k$ below the surface. For waves whose wavelength is short compared with the depth of the water column, the fluid motion is very shallow – hence the name *surface waves*. For waves whose wavelength is commensurate or large compared with h, then the fluid motion spans the entire water column.

Finally, once ϕ is known, note that we can also write

$$\eta(x,y,t) = \int dk_x \int dk_y \hat{\eta}(\boldsymbol{k}) e^{ik_x x + ik_y y - i\omega t} \tag{4.20}$$

where $\hat{\eta}$ is related to $\hat{\phi}$ via the boundary condition at $z = 0$, which becomes

$$-i\omega\hat{\eta} = \left.\frac{d\hat{\phi}}{dz}\right|_{z=0} = A(\boldsymbol{k}) k \sinh(kh) \tag{4.21}$$

so the wave amplitude is

$$\hat{\eta}(\boldsymbol{k}) = \left.\frac{d\hat{\phi}}{dz}\right|_{z=0} = iA(\boldsymbol{k}) \frac{k \sinh(kh)}{\omega} \tag{4.22}$$

Plugging this back, we can finally obtain the actual surface wave displacement $\eta(x,y,t)$.

Example: Suppose we only consider 2D waves, with $k_y = 0$. We also non-dimensionalize the system so $h = 1$ (in which case k_x is expressed in units of $1/h$). Finally, since the amplitude of the wave is arbitrary, we just pick $A = 1$ (this selects its amplitude, and its phase in x). Then, we have that

$$\phi(x,z,t) = \Re\left[A(\boldsymbol{k})\cosh\left(k(z+h)\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}\right] = \cosh\left(k(z+1)\right)\cos(kx-\omega t)$$
$$\eta(x,t) = \Re\left[iA(\boldsymbol{k})\frac{k\sinh(kh)}{\omega}e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}\right] = -\frac{k\sinh(k)}{\omega}\sin(kx-\omega t) \qquad (4.23)$$

where ω depends on k through the dispersion relation. The structure of the solution at time $t = 0$ (and multiples of the wave's period) is shown in Figure 4.2.

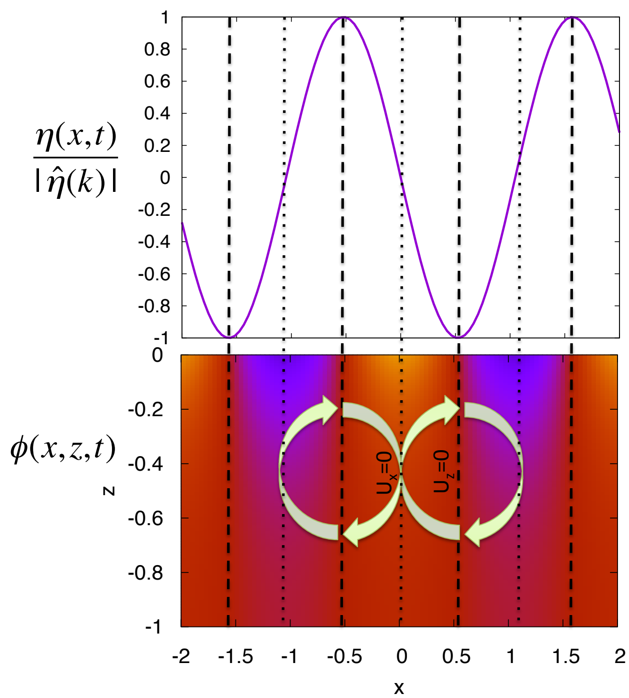

Figure 4.2: Functions $\eta(x,t)$ and $\phi(x,z,t)$ at $t = 0$ for $h = 1$, $k = 3$ and $A = 1$. Note how the peaks and trophs of the waves are regions where ϕ is independent of z which means that $u_z = 0$ (see the vertical black lines). Regions where $\eta = 0$, by contrast, are regions with the largest $|u_z|$ and $u_x = 0$. The corresponding motion is shown in the diagram, and takes the form of ellipses.

Particle paths

As a wave passes by, the surface of the water moves up and down, but as we all know from swimming in the ocean, a body lying in the water moves horizontally as well. To study the particle paths, we apply the same method we did in the case of internal gravity waves. Let's define $\boldsymbol{x}_e(t) = (x_e(t), z_e(t))$ to be the position of a fluid element as a function of time. We have

$$\frac{dx_e}{dt} = u = \frac{\partial \phi}{\partial x} \text{ and } \frac{dz_e}{dt} = w = \frac{\partial \phi}{\partial z} \tag{4.24}$$

For a specific linear mode given by (4.19), and assuming for instance that A is real, we have

$$\begin{aligned} \frac{dx_e}{dt} &= k_x A \cosh\left(k(z_e+h)\right) \Re(ie^{ik_x x_e + ik_y y_e - i\omega t}) \\ &= -k_x A \cosh\left(k(z_e+h)\right) \sin(\boldsymbol{k}\cdot\boldsymbol{x}_e - \omega t) \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \frac{dz_e}{dt} &= Ak \sinh\left(k(z_e+h)\right) \Re(e^{ik_x x_e + ik_y y_e - i\omega t}) \\ &= Ak \sinh\left(k(z_e+h)\right) \cos(\boldsymbol{k}\cdot\boldsymbol{x}_e - \omega t) \end{aligned} \tag{4.26}$$

Solving this equation in general is fairly hard. However, assuming that the displacement around an average position (x_0, y_0) is not too large – that is, assuming $x_e(t) = x_0 + \xi(t)$, and $z_e(t) = z_0 + \zeta(t)$ where ξ and ζ are small, then

$$\begin{aligned} \frac{d\xi}{dt} &\simeq -k_x A \cosh\left(k(z_0+h)\right) \sin(\boldsymbol{k}\cdot\boldsymbol{x}_0 - \omega t) + O(\xi,\zeta) \\ \frac{d\zeta}{dt} &= Ak \sinh\left(k(z_0+h)\right) \cos(\boldsymbol{k}\cdot\boldsymbol{x}_0 - \omega t) + O(\xi,\zeta) \end{aligned} \tag{4.27}$$

This can easily be integrated with time to give

$$\begin{aligned} \xi(t) &\simeq \xi_0 - \frac{k_x}{\omega} A \cosh\left(k(z_0+h)\right) \cos(\boldsymbol{k}\cdot\boldsymbol{x}_0 - \omega t) \\ \zeta(t) &= \zeta_0 - \frac{Ak}{\omega} \sinh\left(k(z_0+h)\right) \sin(\boldsymbol{k}\cdot\boldsymbol{x}_0 - \omega t) \end{aligned} \tag{4.28}$$

This time we see that the particle paths are ellipses. In other words, a wave passing by causes a particle to move around in a vertically aligned ellipse, but does not cause any net displacement. This is illustrated in Figure 4.3. That is, at linear order. Nonlinear effects do cause a net displacement, called *Stokes drift*. The Stokes drift being strongly dependent on the amplitude of the wave, it is much larger near the surface than at the bottom.

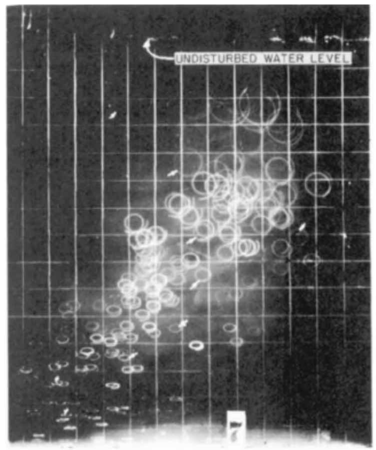

Figure 4.3: Photos of particle paths taken with long exposure, from Wiegel & Johnson (1950). Particles under the surface of the water undergo elliptic motions as the surface wave passes by.

Dispersion relation

The dispersion relation for the surface waves comes from applying the surface boundary condition (4.17) to (4.19) :

$$-\omega^2 \cosh(hk) = -gk \sinh(hk) - \frac{\sigma}{\rho} k^3 \sinh(hk) \tag{4.29}$$

which implies that

$$\omega^2 = k \tanh(hk) \left(g + \frac{\sigma}{\rho} k^2 \right) \tag{4.30}$$

This is the general dispersion relation for surface waves. As usual, we have more than one branch of solution:

$$\omega(\boldsymbol{k}) = \Omega_{\pm}(k) = \pm \sqrt{k \tanh(hk) \left(g + \frac{\sigma}{\rho} k^2 \right)} \tag{4.31}$$

Note how the waves are isotropic along the surface, in the sense that they do not have a preferred direction: ω only depends on the magnitude of k, rather than on k_x and k_y separately. However, these waves are clearly dispersive. The dispersive effects are, this time, quite different from those of internal gravity waves (which caused the group speed and the phase speed to be perpendicular to one another). They will be studied in more detail below.

Capillary waves vs. Gravity waves

We can see that the dispersion relation has two asymptotic limits depending on the size of k:

- if $k \gg \sqrt{g\rho/\sigma}$ (short waves)

$$\Omega_{\pm}(k) \simeq \pm k\sqrt{k\tanh(hk)\frac{\sigma}{\rho}} \tag{4.32}$$

- if $k \ll \sqrt{g\rho/\sigma}$ (long waves)

$$\Omega_{\pm}(k) \simeq \pm k\sqrt{\frac{\tanh(hk)}{kh}gh} \tag{4.33}$$

The first limit describes waves that are dominated by the effects of surface tension, the capillary waves, and the second limit describes waves that are dominated by the effect of gravity, the surface gravity waves. We see that the nature of the wave depends only on their wavelength in comparison with the critical value $2\pi\sqrt{\sigma/\rho g}$. For properties typical of water on Earth, $\sigma \simeq 70$N/m, $g = 10$m/s^2 and $\rho = 1000$kg/m^3, so this critical wavelength is about 0.5 meters, or 50 cm. So any waves whose wavelength is significantly smaller than this is dominated by surface tension, and any wave whose wavelength is significantly larger is dominated by gravity.

When speaking about gravity waves, we sometimes take the further limit of letting $kh \ll 1$, or in other words, to consider waves whose wavelength is large compared with the depth of a layer. This limit is relevant for instance for long-wavelength ocean waves in shallow waters, or even for very-long wavelength (> few km) waves anywhere (since the ocean floor is typically only a few km deep). In that case,

$$\Omega_{\pm}(k) \simeq \pm k\sqrt{gh} \tag{4.34}$$

In that limit, the waves become non-dispersive.

Similarly, when speaking of capillary waves, one often takes the further limit of letting $kh \gg 1$, or in other words, having waves whose horizontal wavelength is much smaller than the depth of the layer considered. That's often the case when we observe capillary waves on the surface of a quiet pond when we are swimming, or tiny ripples generated by water-walking insects. In this more restricted limit, we have

$$\Omega_{\pm}(k) \simeq \pm k\sqrt{k\frac{\sigma}{\rho}} \tag{4.35}$$

In that case the waves remain dispersive.

Phase speed vs. group speed

In general, the phase speed of surface waves is given by

$$c_p = \frac{\omega}{k} = \sqrt{\frac{\tanh(hk)}{k}\left(g + \frac{\sigma}{\rho}k^2\right)} \tag{4.36}$$

so

$$\frac{c_p}{\sqrt{gh}} = \sqrt{\frac{\tanh(\hat{k})}{\hat{k}}\left(1 + \frac{\sigma}{g\rho}\hat{k}^2\right)} \tag{4.37}$$

where $\hat{k} = hk$ is the non-dimensional horizontal wavenumber. The right-hand side tends to 1 for for very small $\hat{k}$ so c_p tends to $\sqrt{gh}$ in this limit. Meanwhile c_p tends to $\sqrt{k\sigma/\rho}$ for very large $\hat{k}$. We therefore see that that phase speed of long surface gravity wave is independent of their wavelength, as for sound waves, but this is not true for capillary waves: longer wavelengths waves travel slower than the shorter wavelengths one, so little ripples in the water travel ahead of the bigger ones.

Note that since the dispersion relation is isotropic, that is, since it only depends on the magnitude of k and not on k_x or k_y explicitly, then the group speed is *in the same direction as* $\boldsymbol{k}$. To see this, note that

$$\boldsymbol{c}_g = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}\right) = \frac{\partial \omega}{\partial k}\left(\frac{\partial k}{\partial k_x}, \frac{\partial k}{\partial k_y}\right) = \frac{\partial \omega}{\partial k}\frac{\boldsymbol{k}}{k} \tag{4.38}$$

This statement is quite general, and applies to any waves whose dispersion relation is independent of the direction of $\boldsymbol{k}$.

The group speed amplitude is then

$$\begin{aligned}
c_g &= \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\sqrt{k\tanh(hk)\left(g + \frac{\sigma}{\rho}k^2\right)} \\
&= \frac{\tanh(hk)\left(g + 3\frac{\sigma}{\rho}k^2\right) + hk\left(g + \frac{\sigma}{\rho}k^2\right)\frac{1}{\cosh^2(hk)}}{2\sqrt{k\tanh(hk)\left(g + \frac{\sigma}{\rho}k^2\right)}} \\
\frac{c_g}{\sqrt{gh}} &= \frac{\tanh(\hat{k})\left(1 + 3\frac{\sigma}{gh^2\rho}\hat{k}^2\right) + \left(1 + \frac{\sigma}{gh^2\rho}\hat{k}^2\right)\frac{\hat{k}}{\cosh^2(\hat{k})}}{2\sqrt{\hat{k}\tanh(\hat{k})\left(1 + \frac{\sigma}{gh^2\rho}\hat{k}^2\right)}}
\end{aligned} \tag{4.39}$$

In the limit $\hat{k} \to 0$ (long waves), the right-hand-side tends to one so we have $c_g \to \sqrt{gh}$ which is also their phase speed. This is not surprising since the waves are non-dispersive in that limit. In the case of $k \to \infty$, we have $c_g \to (3/2)\sqrt{\sigma k/\rho}$.

The two speeds, and their respective limits are shown in the Figure. We see that $c_p > c_g$ for surface gravity waves, while $c_p < c_g$ for capillary waves. This means that the wave crests seem to travel faster than the group for long surface gravity waves, but slower than the group for short capillary waves. Both wave speeds have minima for particular values of kh, although these minima are not at the same position.

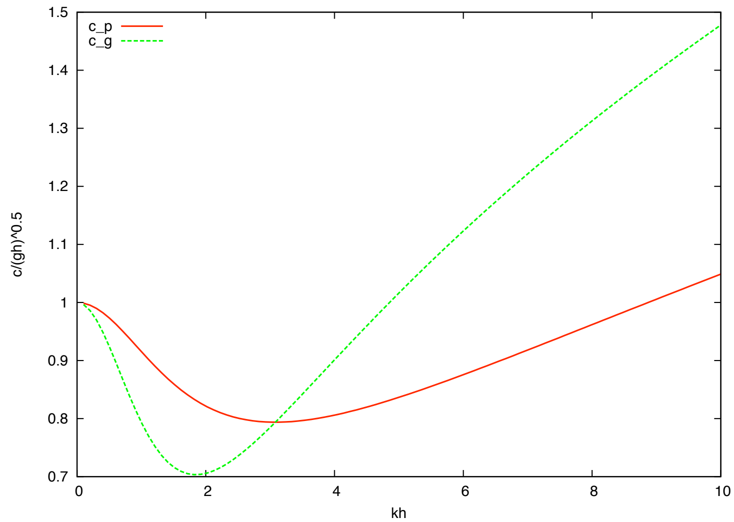

Figure 4.4: Illustrations of the phase and group speeds, for $\frac{\sigma}{gh^2\rho} = 0.1$.