Chapter 3

Incompressibility and the Boussinesq approximation

This Chapter is based on the paper by Spiegel & Veronis, 1960. See also the textbook "Fluid Mechanics" by Kundu & Cohen, pages 124-128.

In all that follows, we will be studying phenomena that occur on timescales that are much longer than the oscillation timescale for sound waves, and where the fluid motions are a lot slower than the sound speed. We would therefore like to find a way of "filtering out" sound waves entirely from our governing equations, to avoid having to discuss them again and again. Since they propagate via compression, we will do so by finding under which conditions a fluid *behaves in an incompressible way*, even when it isn't strictly incompressible. There are in fact a few different ways of doing this. Here, we will treat the simplest case, and derive the so-called *Boussinesq* equations of fluid dynamics (named after Joseph Boussinesq), which provide a very good approximation to the Navier-Stokes equations for water (which is already very nearly incompressible) and also, perhaps surprisingly, astrophysical plasmas in some limit that we will make explicitly clear. The derivation of the Boussinesq approximation for liquids and gases is quite similar up to a point. We now look at both in turn.

3.1 Problem setup

Let's begin by posing the problem. We consider a domain of height H, with unspecified boundary conditions. The domain can be infinitely wide if desired, or have a finite horizontal extent. We then assume that there is a steady background state, with density, pressure and temperature given by

$$\begin{aligned}\bar{\rho}(z) &= \rho_m + \rho_0(z) \\ \bar{p}(z) &= p_m + p_0(z) \\ \bar{T}(z) &= T_m + T_0(z)\end{aligned} \tag{3.1}$$

where the quantities with the subscript m are the vertical means of $\bar{\rho}$, $\bar{p}$ and $\bar{T}$ over the domain, and the functions $\rho_0(z)$, $p_0(z)$ and $T_0(z)$ are the deviations from the mean. The background is assumed to be in mechanical and thermal equilibrium, and satisfies the equation of state, so that

$$\frac{d\bar{p}}{dz} = -\bar{\rho}g$$
$$\frac{d}{dz}\left(k\frac{d\bar{T}}{dz}\right) = -\bar{Q}(z) \tag{3.2}$$

where $\bar{Q}$ is the net heat input into the system (local heating minus local cooling). It also satisfies an equation of state, which we assume to be the perfect gas equation of state for a gas,

$$\bar{p} = R\bar{\rho}\bar{T} \tag{3.3}$$

or

$$\bar{\rho} = \bar{\rho}(\bar{T}) \tag{3.4}$$

for a liquid.

3.2 The incompressibility condition

The idea behind the Boussinesq approximation is to restrict the analysis to that of systems whose background density and temperature do not vary much overall around their mean values[1]. This is an excellent approximation for instance for the ocean, where the density and temperature vary by about 1% and 10% respectively between the bottom and the surface (note that temperature should always be measured in Kelvins). It can *also* be a reasonable approximation for the Earth's atmosphere but only if one considers regions whose heights are much smaller than the density scaleheight (which was around 10km), and that do not host large temperature gradients. Similarly, it is a reasonable approximation in stellar interiors, if one only considers regions whose height is much smaller than the local density and temperature scaleheights. The idea behind that restriction is that if the background state does not vary much, then even if a large scale flow moves a fluid element from the bottom to the top of the domain (and vice versa), the difference between temperature and density in the element and in the ambient fluid will never be very large.

Hence, let's introduce the scaleheights d_f (for the field f being either ρ, T) defined as

$$d_f = f_m \left|\left(\frac{d\bar{f}}{dz}\right)^{-1}\right| = f_m \left|\left(\frac{df_0}{dz}\right)^{-1}\right| \tag{3.5}$$

and let's define the small parameter

$$\epsilon = \frac{H}{\min(d_T, d_\rho)} \tag{3.6}$$

[1]This first step is in fact one of the main differences with the anelastic approximation.

where H is the height of the domain. ϵ is small as long as H is smaller than the smallest of the two scaleheights. We shall therefore assume it is so. Note that, by construction, ϵ is also related to the relative change in the background between the top and the bottom of the domain. Indeed, defining

$$\Delta\rho_0 = |\bar{\rho}(H) - \bar{\rho}(0)| = |\rho_0(H) - \rho_0(0)| \simeq H\left|\frac{d\rho_0}{dz}\right| \simeq \rho_m \frac{H}{d_\rho} \simeq \epsilon\rho_m \tag{3.7}$$

if the smallest scaleheight is d_ρ. If ϵ is constructed from d_T instead, then this should be viewed as an order of magnitude estimate rather than a Taylor expansion, as long as d_ρ and d_T are similar. In the general case, we therefore have $\Delta\rho_0 \sim \epsilon\rho_m$ and $\Delta T_0 \sim \epsilon T_m$.

We now consider fluid motions on this background. We let the velocity field be $\boldsymbol{u}$, and non-dimensionalize it as

$$\boldsymbol{u} = U\hat{\boldsymbol{u}} \tag{3.8}$$

where U is a typical velocity in the fluid.

The flow induces perturbations in ρ, T and p, so that we now have

$$\begin{aligned} \rho(x,y,z,t) &= \bar{\rho}(z) + \tilde{\rho}(x,y,z,t) \\ p(x,y,z,t) &= \bar{p}(z) + \tilde{p}(x,y,z,t) \\ T(x,y,z,t) &= \bar{T}(z) + \tilde{T}(x,y,z,t) \end{aligned} \tag{3.9}$$

Note that if the flow merely moves an element around (without too much compression/expansion), the largest possible perturbation in density or temperature is of the order of the top-to-bottom difference in the field considered. Hence we anticipate that $\tilde{\rho}$ and $\tilde{T}$ are of the order of $\Delta\rho_0$ and ΔT_0 respectively. We therefore non-dimensionalize them as

$$\rho_0(z) + \tilde{\rho} = \Delta\rho_0(\hat{\rho}_0 + \hat{\rho}) \text{ and } T_0(z) + \tilde{T} = \Delta T_0(\hat{T}_0 + \hat{T}) \tag{3.10}$$

where we assume that all the hatted quantities are now of order unity. Finally, we also have to non-dimensionalize the spatial dimensions – and here the obvious lengthscale is H – and time – here, the obvious timescale is H/U (i.e. the domain size divided by the typical velocity). So we set $\boldsymbol{x} = H\hat{\boldsymbol{x}}$ and $t = (H/U)\hat{t}$.

Let's first look at the mass continuity equation. We have:

$$\frac{U}{H}\Delta\rho_0\frac{\partial\hat{\rho}}{\partial\hat{t}} + \frac{U}{H}\left(\rho_m + \Delta\rho_0(\hat{\rho}_0 + \hat{\rho})\right)\hat{\nabla}\cdot\hat{\boldsymbol{u}} + \Delta\rho_0\frac{U}{H}\hat{\boldsymbol{u}}\cdot\nabla\left(\hat{\rho}_0 + \hat{\rho}\right) = 0 \tag{3.11}$$

which implies that

$$\frac{\partial\hat{\rho}}{\partial t} + \left(\frac{1}{\epsilon} + \hat{\rho}_0 + \hat{\rho}\right)\hat{\nabla}\cdot\hat{\boldsymbol{u}} + \hat{\boldsymbol{u}}\cdot\hat{\nabla}\left(\hat{\rho}_0 + \hat{\rho}\right) = 0 \tag{3.12}$$

Keeping only the lowest-order term in ϵ, we find that

$$\hat{\nabla}\cdot\hat{\boldsymbol{u}} = 0 \tag{3.13}$$

This equation is precisely what we were looking for: if $\hat{\nabla} \cdot \hat{\boldsymbol{u}} = 0$, then there is no convergence nor divergence of flows, and in other words, no compression or expansion. Although we already knew this to be the case for liquids, we see that in the Boussinesq approximation, this is also true for gases. Furthermore, if $\hat{\nabla} \cdot \hat{\boldsymbol{u}}$ replaces the mass continuity equation, there is no possibility for sound waves anymore. Note, however, that this is only true to the lowest order in the approximation for gases, and we will need to remember that shortly.

At this point, we now look at the cases of liquids and gases separately.

3.3 The Boussinesq approximation for liquids

Let's look at the equation of state first, going back to dimensional quantities. Taking $\rho = \rho(T)$, and subtracting the background state relationship, we get

$$\frac{\tilde{\rho}}{\rho_m} = \frac{1}{\rho_m}\left(\frac{\partial \rho}{\partial T}\right)_{\bar{\rho},\bar{T}} \tilde{T} \equiv -\alpha \tilde{T} \tag{3.14}$$

which defines α as the *coefficient of thermal expansion*. It is defined with a negative sign, since hotter liquids are usually less dense than cooler ones.

Next, we study the momentum equation. Subtracting the background state hydrostatic equilibrium, it now reads

$$(\rho_m + \Delta\rho_0(\hat{\rho}_0 + \hat{\rho}))\frac{U^2}{H}\frac{\hat{D}\hat{\boldsymbol{u}}}{\hat{D}\hat{t}} = -\frac{1}{H}\hat{\nabla}\tilde{p} - \Delta\rho_0\hat{\rho}g\boldsymbol{e}_z + \frac{U}{H^2}\rho_m\nu\hat{\nabla}^2\hat{\boldsymbol{u}} \tag{3.15}$$

where $\tilde{p}$ was left "as is" because we don't yet know what dimension we expect it to be. Note that we have defined the *viscosity* $\nu = \mu/\rho_m$, and have simplified the viscous stress tensor on the grounds that $\hat{\nabla} \cdot \hat{\boldsymbol{u}} = 0$. On the LHS, one can neglect the terms of order ϵ. Rearranging the equation then yields

$$\frac{\hat{D}\hat{\boldsymbol{u}}}{\hat{D}\hat{t}} = -\frac{1}{\rho_m U^2}\hat{\nabla}\tilde{p} - \frac{gH}{U^2}\frac{\Delta\rho_0}{\rho_m}\hat{\rho}\boldsymbol{e}_z + \frac{\nu}{UH}\hat{\nabla}^2\hat{\boldsymbol{u}} \tag{3.16}$$

The LHS is now explicitly of order unity, and we see that RHS of the non-dimensional momentum equation contains a number of terms, each multiplied by a non-dimensional parameter that describes how important that term is. For the equation to be balanced, at least one of the RHS terms must be of order unity as well.

The non-dimensional term multiplying the viscous force is the inverse of the *the Reynolds number of the flow*, usually defined as

$$Re = \frac{UL}{\nu} \tag{3.17}$$

where U is a characteristic velocity, L a characteristic lengthscale (here, H), and ν is the viscosity. A Reynolds number measures the importance of viscosity in a fluid. When it is much smaller than one, viscosity is very important. When it is much larger than one, viscosity is usually negligible. Since the presence or absence of viscosity has little to do with sound waves, the size of the Reynolds number has little bearing on the Boussinesq approximation – it is what it is.

We see that the term containing gravity (usually called the *buoyancy term*) is multiplied by ϵ times gH/U^2. Note that we can't just ignore it since its size depends on the value of U, which we have not specified yet. The product $\epsilon gH/U^2$ is really the ratio of the potential energy of a typical density perturbation ($\Delta\rho_0 gH$) to its kinetic energy $\rho_m U^2$. For many fluids for which the velocity is large (such as atmospheric jets for instance), this term is often negligible, in which case we may just ignore it. However, in some fluids for which the effect of buoyancy is important (such as convection, or internal waves in the ocean) we really want to keep the buoyancy term, in which case it *must* be of order unity. This implies that the typical velocity scale is expected to be of order

$$U \sim \sqrt{\epsilon g H} \tag{3.18}$$

This is the assumption made in the Boussinesq approximation.

In that case, going back to dimensional quantities we then have

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_m}\nabla\tilde{p} + \frac{\tilde{\rho}}{\rho_m}\boldsymbol{g} + \nu\nabla^2\boldsymbol{u} \tag{3.19}$$

Finally, let's look at the thermal energy equation. Starting with

$$\rho c_v \frac{DT}{Dt} = \nabla\cdot(k\nabla T) + Q + \phi \tag{3.20}$$

subtracting the background state, and linearizing around it by neglecting all terms that are of order ϵ, we have

$$\rho_m c_v \frac{D\tilde{T}}{Dt} + \rho_m c_v w \frac{dT_0}{dz} = \nabla\cdot(k\nabla\tilde{T}) + \tilde{Q} \tag{3.21}$$

where we neglected viscous dissipation (it is usually very small for liquids).

These equations form together the Boussinesq equations for liquids, and are summarized as

$$\begin{aligned} \nabla\cdot\boldsymbol{u} &= 0 \\ \frac{D\boldsymbol{u}}{Dt} &= -\frac{1}{\rho_m}\nabla\tilde{p} - \alpha\tilde{T}\boldsymbol{g} + \nu\nabla^2\boldsymbol{u} \\ \frac{D\tilde{T}}{Dt} + w\frac{dT_0}{dz} &= \frac{1}{\rho_m c_v}\nabla\cdot(k\nabla\tilde{T}) + \frac{\tilde{Q}}{\rho_m c_v} \end{aligned} \tag{3.22}$$

where we have eliminated $\tilde{\rho}$ entirely using the equation of state.

Note that another common form of these equations, assuming that the system is adiabatic *and* that $dT_0/dz = 0$, is

$$\begin{aligned} \nabla\cdot\boldsymbol{u} &= 0 \\ \frac{D\boldsymbol{u}}{Dt} &= -\frac{1}{\rho_m}\nabla\tilde{p} + \frac{\tilde{\rho}}{\rho_m}\boldsymbol{g} + \nu\nabla^2\boldsymbol{u} \\ \frac{D\tilde{\rho}}{Dt} &= 0 \end{aligned} \tag{3.23}$$

where the thermal energy equation was re-cast in terms of $\tilde{\rho}$ using the perturbed equation of state. Because these last equation can also be viewed as an equation for conservation of mass when $\nabla \cdot \boldsymbol{u} = 0$, the following argument is often used to justify the Boussinesq approximation:

" An incompressible fluid has $\nabla \cdot \boldsymbol{u} = 0$, and conservation of mass has $D\rho/Dt = -\rho\nabla\cdot\boldsymbol{u}$ hence in the Boussinesq approximation we must have $D\rho/Dt = 0$."

This argument is wrong, however, and the proper derivation above shows that $D\tilde{\rho}/Dt = 0$ is only true in certain conditions, and is derived strictly from the thermal energy equation rather than mass conservation. The mass conservation equation in the Boussinesq approximation *is* $\nabla \cdot \boldsymbol{u} = 0$.

3.4 The Boussinesq approximation for gases

The derivation of the Boussinesq approximation for gases follows the same steps, but with a few notable differences.

Starting with the equation of state, subtracting the background from $p = R\rho T$, we have

$$\tilde{p} = R(\tilde{T}\bar{\rho} + \tilde{\rho}\bar{T} + \tilde{T}\tilde{\rho}) \tag{3.24}$$

Keeping only the lowest-order terms in ϵ yields

$$\tilde{p} = R(\tilde{T}\rho_m + \tilde{\rho}T_m) \rightarrow \frac{\tilde{p}}{p_m} = \frac{\tilde{T}}{T_m} + \frac{\tilde{\rho}}{\rho_m} \tag{3.25}$$

Although this has not been explicitly written out, the RHS clearly looks like it should be of order ϵ.

Next, we study the momentum equation. Subtracting the background state hydrostatic equilibrium, it now reads

$$(\rho_m + \Delta\rho_0(\hat{\rho}_0 + \hat{\rho}))\frac{U^2}{H}\frac{\hat{D}\hat{\boldsymbol{u}}}{\hat{D}\hat{t}} = -\frac{1}{H}\hat{\nabla}\tilde{p} - \Delta\rho_0\hat{\rho}g\boldsymbol{e}_z + \frac{U}{H^2}\mu\hat{\nabla}^2\hat{\boldsymbol{u}} \tag{3.26}$$

where $\tilde{p}$ again was left "as is". By contrast with the case of liquids, however, $\tilde{p}$ is not independent of $\tilde{T}$ and $\tilde{\rho}$ – but instead is related to them through the perturbed equation of state.

As before, on the LHS we neglect the terms of order ϵ. To study the RHS, first note that from hydrostatic equilibrium we have

$$\frac{1}{p_m}\left|\frac{dp_0}{dz}\right| = \frac{1}{d_p} = \frac{\rho_m}{p_m}g = \frac{\gamma}{c^2}g \tag{3.27}$$

where d_p is the pressure scaleheight, and where we have defined c to be the speed of sound associated with the mean thermodynamical state:

$$c^2 = \gamma R T_m \tag{3.28}$$

where γ is the adiabatic index (see Chapter 2). Hence

$$g = \frac{c^2}{\gamma d_p} \tag{3.29}$$

Substituting this into the momentum equation together with the linearized equation of state, keeping only the lowest-order terms in ϵ, and simplifying somewhat, we have

$$\frac{\hat{D}\hat{\boldsymbol{u}}}{\hat{D}\hat{t}} = -\frac{1}{M^2\gamma}\hat{\nabla}\left[\frac{\Delta T_0}{T_m}\hat{T} + \frac{\Delta\rho_0}{\rho_m}\hat{\rho}\right] - \frac{\Delta\rho_0}{\rho_m}\frac{1}{M^2\gamma}\frac{H}{d_p}\hat{\rho}\boldsymbol{e}_z + \frac{\nu}{HU}\hat{\nabla}^2\hat{\boldsymbol{u}} \tag{3.30}$$

where $M \equiv U/c$ is defined to be the *Mach number* of the flow.

To model systems which are not just decaying viscously, either the pressure term or the buoyancy term must be sufficiently large. We see that the pressure term contains the quantity $\epsilon/M^2\gamma$, while the buoyancy term contains the quantity $(H/d_p)\epsilon/M^2\gamma$. The relative importance of the two terms depends somewhat on the size of H/d_p.

Since we are considering a perfect gas, d_p is usually of the same order as d_ρ and d_T (within a factor of order unity). Hence H/d_p is in fact of order ϵ. The buoyancy term then appears to be much smaller than the pressure term. While this may be ok for some types of flows (as discussed in the previous section for liquids), it cannot be true if we are studying phenomena that intrinsically depend on the buoyancy (such as convection, internal gravity waves, etc). The solution to this conundrum is to realize that the pressure term does not have to be of order $\epsilon/M^2\gamma$: if we require, in addition, that

$$\frac{\Delta T_0}{T_m}\hat{T} \simeq -\frac{\Delta\rho_0}{\rho_m} \rightarrow \frac{\tilde{T}}{T_m} \simeq -\frac{\tilde{\rho}}{\rho_m} \tag{3.31}$$

to order H/d_p, that is, if we neglect the pressure perturbations entirely in the equation of state, then the buoyancy term and the pressure term are of the same order, $(H/d_p)\epsilon/M^2\gamma$. The requirement that the forcing be of order one (to have any effect on the acceleration of the fluid) sets the Mach number to be of order ϵ – in other words, the Boussinesq approximation will only be valid for flows whose velocity is much smaller than the sound speed.

Finally, let's look at the thermal energy equation. We start with the complete dimensional equation:

$$\rho c_v \frac{DT}{Dt} = -p\nabla\cdot\boldsymbol{u} + \nabla\cdot(k\nabla T) \tag{3.32}$$

where we have neglected the viscous heating term (on the grounds that it is negligible), and assumed the internal heating/cooling is zero. Note that we have to keep the term $p\nabla\cdot\boldsymbol{u}$ on the RHS, because $\nabla\cdot\boldsymbol{u}$ is of order ϵ, which is the same order as the term in the LHS. We have:

$$p\nabla\cdot\boldsymbol{u} \simeq -\frac{p_m}{\rho_m}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)(\bar{\rho}(z) + \tilde{\rho}) \tag{3.33}$$

Using the background equation of state to replace $\bar{\rho}$, and (3.31) to eliminate $\tilde{\rho}$, we have

$$p\nabla \cdot \boldsymbol{u} \simeq \frac{p_m}{\rho_m}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\left(\rho_m \frac{\tilde{T}}{T_m} - \frac{\bar{p}}{R\bar{T}}\right) \tag{3.34}$$

Since $\Delta p_0 \sim \epsilon p_m$, we can rewrite the RHS keeping only the lowest-order terms in ϵ:

$$p\nabla \cdot \boldsymbol{u} \simeq \left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\left(p_m \frac{T_0(z)+\tilde{T}}{T_m} - p_0(z)\right) = \frac{p_m}{T_m}\frac{D}{Dt}(T_0(z)+\tilde{T}) - w\frac{dp_0}{dz} \tag{3.35}$$

Combining this with the thermal energy equation, keeping only the lowest-order terms in ϵ, and using hydrostatic equilibrium to replace dp_o/dz, we get

$$\rho_m c_p \frac{D}{Dt}(T_0(z)+\tilde{T}) + wg\rho_m = \nabla\cdot(k\nabla\tilde{T}) \tag{3.36}$$

where we introduced the constant

$$c_p = c_v + R = c_v + \frac{p_m}{\rho_m T_m} \tag{3.37}$$

This constant is the *specific heat at constant pressure*. It can be shown using thermodynamics that the adiabatic index γ is also the ratio of specific heats:

$$\gamma = \frac{c_p}{c_v} \tag{3.38}$$

The thermal energy equation can then be rewriten as

$$\frac{D\tilde{T}}{Dt} + w\left(\frac{dT_0}{dz} + \frac{g}{c_p}\right) = \frac{1}{\rho_m c_p}\nabla\cdot(k\nabla\tilde{T}) \tag{3.39}$$

And finally, note that $-g/c_p$ is the *adiabatic temperature gradient*, that is, the temperature gradient that would be present if the background was adiabatically stratified. To see this, note that we have in general

$$\frac{1}{T_m}\frac{dT_0}{dz} = \frac{1}{p_m}\frac{dp_0}{dz} - \frac{1}{\rho_m}\frac{d\rho_0}{dz} \tag{3.40}$$

from the linearized equation of state. If the background is adiabatically stratified, then

$$\frac{1}{\rho_m}\frac{d\rho_{\rm ad}}{dz} = \frac{1}{\gamma p_m}\frac{dp_{\rm ad}}{dz} \tag{3.41}$$

so

$$\frac{1}{T_m}\frac{dT_{\rm ad}}{dz} = (1-\gamma^{-1})\frac{1}{p_m}\frac{dp_{\rm ad}}{dz} = -\left(1-\frac{c_v}{c_p}\right)\frac{\rho_m}{p_m}g = -\frac{R}{c_p}\frac{\rho_m}{p_m}g \tag{3.42}$$

Hence the adiabatic temperature gradient is $dT_{\rm ad}/dz = -g/c_p$.

To summarize, the Boussinesq equations for a gas are:

$$\nabla \cdot \boldsymbol{u} = 0$$
$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_m}\nabla\tilde{p} + \frac{\tilde{\rho}}{\rho_m}\boldsymbol{g} + \nu\nabla^2\boldsymbol{u}$$
$$\frac{\tilde{T}}{T_m} = -\frac{\tilde{\rho}}{\rho_m}$$
$$\frac{D\tilde{T}}{Dt} + w\left(\frac{dT_0}{dz} - \frac{dT_{\rm ad}}{dz}\right) = \nabla \cdot \left(\frac{k}{\rho_m c_p}\nabla\tilde{T}\right) \tag{3.43}$$

The quantity $k/\rho c_p$ is the *thermal diffusivity* and is often noted as κ_T. Note that the perturbed equation of state can also be cast as $\tilde{\rho} = -\alpha\rho_m\tilde{T}$ exactly as in the case of liquids. We see that the only difference with the case of a liquid is in the thermal energy equation, which contains the additional adiabatic temperature gradient, and where c_v was replaced by c_p.

3.5 Discussion and things to remember

In this Chapter, we learned about the Boussinesq approximation, which provides a means of studying the dynamics of fluids without necessarily having to resolve sound waves (the latter are effectively filtered out). We saw that it is valid whenever the domain considered is much shorter than a density or temperature scaleheight (whichever is smaller). The Boussinesq equations for liquids and gases are essentially similar, except for the thermal energy equation.

Other equations exist that also filter out sound waves, and that are valid in a broader class of problems, notably those where the domain is much larger than a density/temperature scaleheight. These are based on the requirement that the velocity of the fluid be much smaller than the sound speed. Among those are the *anelastic approximation* and the *pseudo-incompressible approximation*. For more information on the latter, as well as another, possibly much more elegant derivation of the Boussinesq approximation, see the paper by Vasil et al. 2013.