Homework 4

February 2, 2018

1 Diffraction of sound waves

Consider a 2D uniform medium (with constant sound speed). Then, consider a small region where the sound speed is slightly larger, i.e. by writing that

$$c_s(X,Y) = c_0 + (\Delta c)\exp\left[-\frac{(X^2+Y^2)}{2}\right] \quad (1)$$

where $\Delta c > 0$.

What will happen to the paths of waves coming from $Y \to +\infty$, along wavevectors that are initially along $\boldsymbol{e}_y$, as they approach the sound speed anomaly? A qualitative answer is sufficient. Sketch some representative ray paths. What happens if the sound speed anomaly Δc is negative? Sketch the ray paths in that case.

2 2D world

If the world were a disk, and we lived in 2D, what would be rate of decay of the amplitude of sound away from a *circular* loudspeaker ?

In what follows, do Question 3 OR Question 4.

3 Global modes with non-constant c

Consider a model setup as in Section 2.6, with a linearly varying sound speed $c(z)$ below a surface at $z = 0$. Consider that the system is periodic in the x direction. Solve for the global modes of oscillation, i.e. use separation of variables to solve for the global eigenmodes of the wave equation with $c(z) = c_0 + c_1 z$. Assume that $p = 0$ at the surface, is periodic in x, and assume that there is a lower boundary at $z = H$ where $p = 0$ too. You may have to solve this problem numerically if you cannot find an analytical solution. Compare (qualitatively) the structure of the normal modes to the wave packet solutions. What similarities do you see?

4 Gravity waves wave packet equations – the hard way

Consider the governing equation for internal gravity waves (equation 4.7), in the case where N is a slowly varying function of $\boldsymbol{X}$ and the slow time τ. Assume a wave packet solution of the form

$$\phi = A(\boldsymbol{X}, \tau) e^{i\theta(\boldsymbol{x},t)} \tag{2}$$

with $\boldsymbol{k} = \nabla\theta$ and $\omega = -\partial\theta/\partial t$.

Starting from these assumptions only, directly prove equations (4.23), (4.28), (4.29) of the lecture notes. These are not too difficult.

Then if you really have time, and for extra credit, try to prove (4.30). This last one is quite hard. You may want to first show that the amplitude equation satisfies:

$$2\omega\frac{\partial k^2}{\partial\tau}A + k^2\frac{\partial\omega}{\partial\tau}A + 2\omega k^2\frac{\partial A}{\partial\tau} - \omega^2\left(2\boldsymbol{k}\cdot\nabla A + A\nabla\cdot\boldsymbol{k}\right) = -\left[N^2\frac{\partial k_x}{\partial X}A + k_x\frac{\partial N^2}{\partial X}A + 2k_xN^2\frac{\partial A}{\partial X}\right] \tag{3}$$

Then, using the standard trick of multiplying by the complex conjugate of A, and rearranging terms, write this equation as

$$\frac{\partial}{\partial\tau}(|A|^2k^2) + \nabla\cdot\left(\boldsymbol{c}_g k^2|A|^2\right) = \text{stuff} \tag{4}$$

The whole problem then reduces to showing that "stuff" is the RHS of (4.30). Equation (4.24) will turn out to be very useful.