Homework 3 on Chapter 2

January 24, 2018

1 Global modes in a square

Find the 2D eigenmodes and eigenvalues of the wave equation $\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$ in a square whose side length is 1, subject to $p = 0$ on the boundary of the region. Plot a few representative eigenmodes in each case. What are all the possible frequencies achievable?

2 Multiscale expansion for the damped oscillator

Problem 1:

- Consider the function $f(x) = e^{-\epsilon x} \sin(x)$. Calculate its derivative $\frac{df}{dx}$.
- Write $f(x)$ as a function of two variables $X_s = \epsilon x$ (the slow space variable), and $X_f = x$ (the fast space variable) as $f(X_s, X_f) = e^{-X_s} \sin(X_f)$. Compute

$$\frac{df}{dx} = \frac{\partial f}{\partial X_s}\frac{\partial X_s}{\partial x} + \frac{\partial f}{\partial X_f}\frac{\partial X_f}{\partial x} \tag{1}$$

  and verify that you recover the same answer answer as in the first case.

Problem 2: Consider the ODE

$$\frac{d^2 f}{dt^2} + f = -\epsilon \frac{df}{dt} \tag{2}$$

with initial conditions $f(0) = 1$ and $\frac{df}{dt}(0) = 0$. Write $f(t)$as the function $f(T_s, T_f)$ where $T_s = \epsilon t$ is the slow time and $T_f = t$ is the fast time. Compute $\frac{df}{dt}$, and $\frac{d^2 f}{dt^2}$, and use these to solve the ODE approximately, order-by-order in ϵ. Hint: you will find the answer (or close to the answer) in the AMS 212B lecture notes (see link in the Lecture page).

3 Wave packet approximation

Problem 1: Prove equation (2.110) of the online notes using (2.105), (2.107), and (2.109).

Problem 2: Prove equation (2.122) and (2.123) of the lecture notes.

4 Application : A sound mirror

At the San Francisco Exploratorium, there is a rather amazing example of a sound mirror. The display is composed of two walls facing each other at two ends of a (usually very crowded) room, and each wall has a 2m-size half-sphere carved into it. When one person stands at the edge of the half-sphere and speaks into it, even very gently, another person standing in the same position but near the other half-sphere can hear the first person's voice very clearly. If, on the other hand, the first person turns around and speaks directly towards the second, it's usually impossible for the two people to hear each other across the noisy room.

Explain how the sound mirror works, and why speaking directly to one another wouldn't. Hint: use ray theory and assume that a half-sphere is not too different from a parabola for simplicity. A parabola has the property that parallel rays coming in from infinity all focus in the same point (called the focal point). Similarly, rays emitted from the focus end up going to infinity on parallel lines, after reflection on the mirror. See Wikipedia article on Parabolic Reflectors for instance.

Note: there are two levels of answer to this question: a qualitative, pictorial answer, and a quantitative one. Try the qualitative first, and if you really understand what you are doing, then the quantitative.

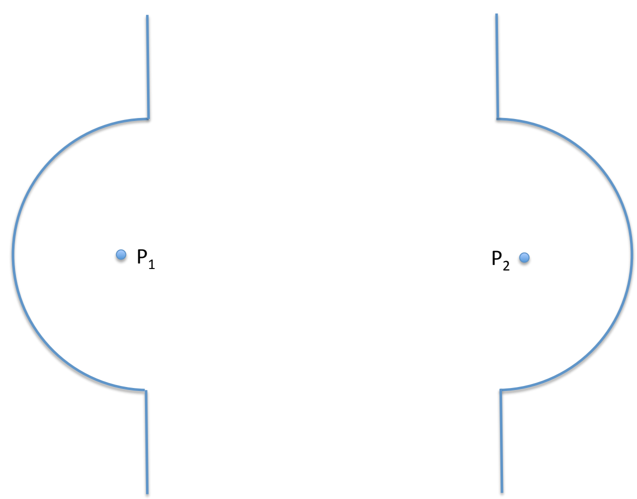

Figure 1: Schematic of the sound mirror. The two people are sitting at the two foci of the mirrors at P_1 and P_2, talking to each other while facing the mirrors, can hear each other across a crowded, noisy room.