Homework 1 (from Chapter 1)

1 Mass conservation and other conservation laws

1.1 Mass conservation in polar coordinates

Question 1: Consider the mass conservation equation, and express it in 2D polar coordinate (r, θ).

Question 2: Assume that the velocity field $\boldsymbol{u} = (u_r, u_\theta) = (u_0, 0)$, where u_0 is constant. Sketch this velocity field.

Question 3: Solve the mass conservation equation for the density $\rho(r, t)$ (you can assume, or prove that, the solution is independent of θ), with the the velocity field above, assuming that at $t = 0$ $\rho(r, \theta, t = 0) = r \exp(-(r-1)^2/2)$. Plot the results, and discuss your findings. Hint: mathematically, it may be easier to consider the function $f(r, t) = r\rho(r, t)$.

1.2 Other conservation laws

Assume that a fluid is known to transport a scalar quantity S in a strictly conserved way. What is the equation of evolution of the scalar S?

2 Momentum equation and the vorticity equation

Question 1: Assuming that the flow is incompressible, and that kinematic viscosity μ is constant, show that $\nabla \cdot \boldsymbol{\Pi} = \mu \nabla^2 \boldsymbol{u}$ for a Newtonian fluid (i.e. a fluid where $\boldsymbol{\Pi}$ is given by equations 1.21 or equivalently 1.22 in the notes). Hint: you may express the tensor in Cartesian coordinates first.

Question 2: What is the expression for $\nabla \cdot \boldsymbol{\Pi}$ when μ is *not* constant?

Question 3: Assuming that $\nabla \cdot \boldsymbol{\Pi} = \mu \nabla^2 \boldsymbol{u}$, derive the equation for the evolution of vorticity $\omega = \nabla \times \boldsymbol{u}$ from the momentum equation, assuming that density is constant. Show that if viscosity if negligible, then vorticity is conserved in 2D but not in 3D. Hint: that last question requires (1) using the formulary provided to rewrite $\nabla \times (\boldsymbol{u} \cdot \nabla \boldsymbol{u})$ as the sum of an advected term and a vortex stretching term, and (2) thinking about the direction of the vorticity vector in 2D.

3 The equation of state, the thermal energy equation and hydrostatic equilibrium

Hydrostatic equilibrium is derived from the momentum equation assuming there are no fluid motions. In that case there is an exact balance between the pressure gradient and the gravitational force, namely

$$-\nabla p + \rho \boldsymbol{g} = 0 \tag{1}$$

Question 1: Express this equation in a Cartesian coordinate system, assuming that the gravity $\boldsymbol{g} = -g\boldsymbol{e}_z$.

Question 2: Consider a liquid equation of state, and assume the liquid is isothermal (e.g. fresh water resting on a flat surface in a container). Express hydrostatic equilibrium as an equation for the pressure. Solve this equation from $z = 0$ (the ground) upward until the water surface at $z = H$, assuming that $p = 0$ at the surface. Interpret your solution.

Question 2: Assume instead that the fluid is a gas, satisfying a perfect gas equation of state, and it is isothermal. Express hydrostatic equilibrium as an equation for the density only. Solve this equation from $z = 0$ upward, assuming that $\rho = \rho_0$ at $z = 0$. Interpret your solution. What is the density scale-height?

Question 4: Assume instead that the fluid is adiabatic, with adiabatic index γ. Express hydrostatic equilibrium as an equation for the density only. Solve this equation from $z = 0$ upward, assuming that $\rho = \rho_0$ at $z = 0$. How do the solutions change with γ? Interpret your solution.