5.3.4 Truncated equations and the Lorentz system

As a final foray into nonlinear theory, we now explore the idea of using *truncated systems* as a tool to study the time-dependent dynamics of convection close to onset. Indeed, as we saw previously, only very few modes are excited near onset. It may then be possible to model the system dynamics by considering the nonlinear interaction between these few modes only. This procedure will also verify our assumption of the previous section that the steady states found are meaningful representations of the near-onset dynamics.

Derivation of the Lorenz equation

Our previous calculation of the weakly nonlinear *steady-state* just above onset suggests that the modes that matter for small Ra are:

$$\begin{aligned} \phi(x,z,t) &= A(t)\sin(\pi z)\cos(k_c x) \\ T(x,z,t) &= B(t)\sin(\pi z)\sin(k_c x) - C(t)\sin(2\pi z) \end{aligned} \tag{5.1}$$

Plugging these expressions into (6.36), we get

$$\omega = -\nabla^2\phi = (\pi^2 + k_c^2)\phi \tag{5.2}$$

and so the momentum equation (6.37) becomes (recalling that the nonlinear term in the momentum equation is zero)

$$\begin{aligned} (\pi^2 + k_c^2)\dot{A}\sin(\pi z)\cos(k_c x) &= \mathrm{RaPr}Bk_c\sin(\pi z)\cos(k_c x) \\ &- \mathrm{Pr}(\pi^2 + k_c^2)^2 A\sin(\pi z)\cos(k_c x) \end{aligned} \tag{5.3}$$

which simplifies directly to

$$\dot{A} = \mathrm{RaPr}\frac{k_c}{\pi^2 + k_c^2}B - \mathrm{Pr}(\pi^2 + k_c^2)A \tag{5.4}$$

The temperature equation (6.39) on the other hand becomes

$$\begin{aligned} &\dot{B}\sin(\pi z)\sin(k_c x) - \dot{C}\sin(2\pi z) + k_c\pi B\sin(\pi z)\cos(k_c x)A\cos(\pi z)\cos(k_c x) \\ &+ [\pi B\cos(\pi z)\sin(k_c x) - 2\pi C\cos(2\pi z)]\, k_c A\sin(\pi z)\sin(k_c x) \\ &- k_c A\sin(\pi z)\sin(k_c x) \\ &= -(\pi^2 + k_c^2)B\sin(\pi z)\sin(k_c x) + 4\pi^2 C\sin(2\pi z) \end{aligned} \tag{5.5}$$

Simplifying this, we get:

$$\begin{aligned} &\dot{B}\sin(\pi z)\sin(k_c x) - \dot{C}\sin(2\pi z) + \frac{1}{2}k_c\pi AB\sin(2\pi z) \\ &- \pi k_c AC\sin(k_c x)\left(\sin(3\pi z) - \sin(\pi z)\right) - k_c A\sin(\pi z)\sin(k_c x) \\ &= -(\pi^2 + k_c^2)B\sin(\pi z)\sin(k_c x) + 4\pi^2 C\sin(2\pi z) \end{aligned} \tag{5.6}$$

We see that there are 3 types of terms: terms in $\sin(\pi z)$, in $\sin(2\pi z)$ and in $\sin(3\pi z)$. Projecting this equation onto $\sin(\pi z)$ and $\sin(2\pi z)$ respectively (i.e.

integrating this equation times $\sin(\pi z)$ or $\sin(2\pi z)$ from $z = 0$ to $z = 1$), and ignoring the term in $\sin(3\pi z)$, we then get

$$\begin{aligned}
\dot{A} &= \mathrm{RaPr}\frac{k_c}{\pi^2 + k_c^2}B - \mathrm{Pr}(\pi^2 + k_c^2)A \\
\dot{B} &= k_c A - k_c \pi AC - (\pi^2 + k_c^2)B \\
\dot{C} &= \frac{1}{2}k_c \pi AB - 4\pi^2 C
\end{aligned} \tag{5.7}$$

With a little bit of work, it is then possible to rescale these equations into

$$\begin{aligned}
a' &= \mathrm{Pr}(-a + b) \\
b' &= ra - b - ac \\
c' &= -sc + ab
\end{aligned} \tag{5.8}$$

where $r = \mathrm{Ra}/\mathrm{Ra}_c$, a is proportional to A, b is proportional to rB, c is proportional to rC, and time has been rescaled as well (so the derivative with respect to the new time is denoted by a prime instead of a dot). s is merely a constant that depends on the time-non-dimensionalization. It is traditionally taken to be 8/3. The new set of equation is very famous: they form *the Lorenz equations.* Note how a represents the amplitude of the convective rolls, b represents the amplitude of the corresponding temperature perturbation, and c corresponds to the change in the horizontally-averaged temperature profile.

Properties of the Lorenz equations

The Lorenz equations have been studied in depth, and their discovery started the field of *chaotic dynamics.* Let's briefly look into their propoerties. First, note that they have an obvious fixed point at $a = b = c = 0$ (which corresponds to the state of no convection). Do they have other fixed points? If they do, then the latter must satisfy $a = b$, and $c = ab/s = a^2/s$, and so

$$ra - a - a^3/s = 0 \tag{5.9}$$

Aside from the $a = 0$ steady-state which we already know of, we see there are two other solutions with $a = \pm s(r-1)$. These solutions only exist when $r \geq 1$, so $r = 1$ marks an important bifurcation in the system. We know this bifurcation, of course – it is the one that corresponds to $\mathrm{Ra} = \mathrm{Ra}_c$. We see that, for r below 1, the fixed point at the origin is stable. For r slightly above 1, the fixed point at the origin is unstable, but there are two new *stable* steady convective states. This justifies the approach selected in the previous lecture.

Further investigation shows that (at least in the Lorenz system), these new fixed points also become unstable at the critical value

$$r = \frac{\mathrm{Pr}(\mathrm{Pr} + s + 3)}{\mathrm{Pr} - s - 1} \tag{5.10}$$

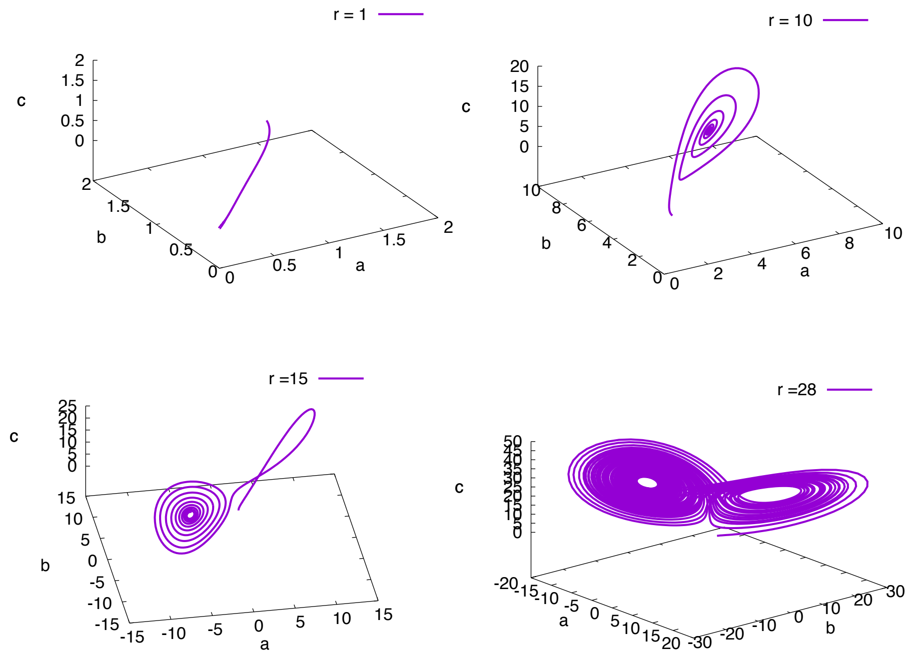

Figure 5.1: Evolution of the Lorenz system for $r = 0.1$, $r = 10$, $r = 15$ and $r = 28$. The first case is subcritical, and the fixed point at the origin is globally stable. The second and third cases, the fixed point at the origin is unstable, but two new fixed points emerge. In the final case all the fixed points are unstable, and the system evolves on a chaotic attractor.

Beyond that point, *the solutions become chaotic and converge to a strange attractor*. This means that they are strictly *not* periodic, and strictly *not* predictable beyond a certain timescale, but nevertheless have somewhat recognizable patterns. These various dynamical behaviors are illustrated in Figure 5.1, for $\mathrm{Pr} = 10$ and $s = 8/3$. The bifurcation parameter r is varied from 0.1 to 28.

5.4 Discussion and things to remember

In this Chapter we explored 4 different ways of studying an instability: a local linear stability analysis (which, in this case, did not work), a global linear stability analysis, an energy stability analysis, and finally, different ways of approaching the question of weakly nonlinear stability. These techniques can be applied to any instability study in similar ways, although the details and the outcomes will of course vary.

For instance, there are many instabilities for which a local analysis is indeed very helpful, as in the case of double-diffusive convection, for instance. Meanwhile, there are many systems for which energy stability as introduced here will not give a particularly useful answer – as in the case of stratified shear flows. Finally, weakly nonlinear theory, as we saw, can be quite difficult to investigate – and the case studied here was one of the simpler examples!

If all else fails, of course, it is often possible to study the problem experiementally – either using laboratory experiments, or using numerical experiments.