5.3.3 An introduction to weakly nonlinear theory

Weakly nonlinear theory is the study of the dynamics of a system that is only weakly nonlinear, that is, a system where the amplitude of the perturbations is just large enough for the nonlinear terms to become relevant. This happens when the control parameter (in our case, Ra) is close to the critical value Ra_c, in which case this also means that there are usually only very few unstable modes. The idea is to create a reduced set of equations that describes the nonlinear interaction between these few modes only.

There are a number of different ways of constructing weakly nonlinear equations – the procedure is not unique, and neither is the result. However, only one way of studying the problem will actually yield equations whose behavior *correctly* models observations. This is the main reason why weakly nonlinear theory is so difficult to approach at first, and somewhat frustrating to learn. However, with experience and practice, things get a lot easier. The key is to already have an idea, ahead of time, of the types of dynamics we need to end up with (based on experiments, or numerical simulations), and use this information to find our way more directly.

Before we start, however, a small mathematical detour (that will make our lives infinitely nicer later) is necessary.

Mathematical detour: Fredholm's alternative and the solvability condition

See textbook "Applied Partial Differential Equations" by Haberman, for more.

Consider the following ODE problem:

$$\mathcal{L}u(x) = F(x) \tag{5.1}$$

on the interval $[a, b]$, subject to homogeneous boundary conditions (that is, $u(a) = u(b) = 0$). The operator $\mathcal{L}$ is assumed non-singular, and so is the forcing $F(x)$.

Fredholm's alternative states that if $\mathcal{L}$ is a self-adjoint operator for the inner product $\langle\cdot\rangle$ (see definition below) and if there exists a solution to the homogeneous equation

$$\mathcal{L}u_h(x) = 0 \tag{5.2}$$

where $u_h(x)$ is not identically 0, then the original problem $\mathcal{L}u(x) = F(x)$ has a solution only if

$$\langle F, u_h\rangle = 0 \tag{5.3}$$

This is called a *solvability condition.* Note that there is actually more to Fredholm's alternative, but for the purpose of what we are about to do, this is all we need to know. See the textbook listed above for more detail.

Let's now recast this jargon in slightly less obscure terms, and actually prove that this statement is correct. First, note that a self-adjoint operator $\mathcal{L}$ is one for which there exists a weight function $r(x)$, and a corresponding inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)r(x)dx \tag{5.4}$$

such that

$$\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle \tag{5.5}$$

Note that the inner product is symmetric, so we always have $\langle f, g\rangle = \langle g, f\rangle$ for all f, g.

If $\mathcal{L}$ is self-adjoint with that inner product, then it's fairly easy to show that different solutions v_n to the *eigenvalue problem*

$$\mathcal{L}v_n(x) = \lambda_n v_n(x) \tag{5.6}$$

are orthogonal if they have different eigenvalues λ_n. Indeed,

$$\langle v_m, \mathcal{L}v_n\rangle = \lambda_n \langle v_m, v_n\rangle \tag{5.7}$$

but also, since $\mathcal{L}$ is self-adjoint, then

$$\langle v_m, \mathcal{L}v_n\rangle = \langle \mathcal{L}v_m, v_n\rangle = \lambda_m \langle v_m, v_n\rangle \tag{5.8}$$

This can only be true if $\langle v_m, v_n\rangle = 0$ since we assumed that $\lambda_n \neq \lambda_m$. Similarly, it can be shown that the eigenvalues λ_n have to be real.

It is then also possible to show that while any two eigenfunctions v_n and v_m with the same eigenvalue λ are not necessarily orthogonal, it is possible to find linear combinations of the latter that *are* orthogonal. This process is called Gram-Schmidt orthogonalization. The eigenfunctions (whose amplitudes are always arbitrary) can also be normalized so that $\langle v_n, v_n\rangle = 1$.

Finally, with this set of *orthonormal* eigenfunctions, it is also possible (but not very easy) to show that *any* function $f(x)$ defined on the interval $[a, b]$ and satisfying the homogeneous boundary conditions $f(a) = f(b) = 0$ can be written as

$$f(x) = \sum_n f_n v_n(x) \tag{5.9}$$

This is often called a *generalized Fourier expansion*.

What does this buy us? Well, we can now use this information in our original problem, and let its solution $u(x)$ be

$$u(x) = \sum_n a_n v_n(x) \tag{5.10}$$

We then have

$$\mathcal{L}u(x) = \sum_n a_n \mathcal{L}v_n(x) = \sum_n a_n \lambda_n v_n(x) = F(x) \tag{5.11}$$

Taking the dot-product of this equation with v_m, we then get

$$\sum_n a_n \lambda_n \langle v_m, v_n\rangle = a_m \lambda_m = \langle F, v_m\rangle \tag{5.12}$$

As long as none of the λ_m are zero, this can be solved with

$$a_m = \frac{\langle F, v_m \rangle}{\lambda_m} \tag{5.13}$$

and yields a unique solution for $u(x)$.

However, *if one of the* λ_m *is zero and the projection of* F *on its corresponding eigenmode* v_m *is not* ($\langle F, v_m \rangle \neq 0$), then there is no solution to this equation, because (5.83) is contradictory. For a solution to exist, we must have $\langle F, v_m \rangle = 0$ for the eigenmode v_m whose eigenvalue is 0.

In our original problem, we are told that there is a non-zero solution to

$$\mathcal{L} u_h = 0 \tag{5.14}$$

so in other words, u_h is an eigenmode of $\mathcal{L}$ with eigenvalue 0. We are exactly in the conditions of the theorem, and therefore know that $\mathcal{L}u(x) = F(x)$ only has a solution if $\langle F, u_h \rangle = 0$.

These considerations can easily be generalized to operators acting in multiple spatial dimensions, and operating on more than 1 function. This will come in *very* handy later.

A simple example of application of weakly nonlinear theory

This section is adapted from Chapter 5.2 of the textbook "Introduction to hydrodynamic stability" by Drazin.

Let's consider a toy problem that nicely illustrates how one may study weakly nonlinear stability. Consider the following equation that contains a diffusion term, and a nonlinear term:

$$\frac{\partial u}{\partial t} - \sin u = \frac{1}{R}\frac{\partial^2 u}{\partial z^2} \tag{5.15}$$

with boundary condition $u = 0$ at $z = 0$ and $z = \pi$. This equation has a simple null solution, $u = 0$, which forms the background around which we will linearize. In what follows, we therefore assume u is small. Using the fact that $\sin u \simeq u$ for small enough u, the linearized equation is

$$\frac{\partial u}{\partial t} - u = \frac{1}{R}\frac{\partial^2 u}{\partial z^2} \tag{5.16}$$

Seeking solutions of the kind

$$u(z,t) = \hat{u}(z)e^{\lambda t} \tag{5.17}$$

we find that $\hat{u}$ satisfies

$$\frac{d^2 \hat{u}}{dz^2} = R(\lambda - 1)\hat{u} \tag{5.18}$$

which, given the required boundary conditions, has solutions $\hat{u}_n \propto \sin(nz)$ with $\lambda_n = 1 - n^2/R$. For λ_n to be positive, R must be greater than n^2. Hence the most unstable mode will be $n = 1$, and the critical value of R above which this mode is excited is $R_c = 1$. For R above R_c (but smaller than 4), the only mode excited is the $n = 1$ mode.

Suppose $R = 1 + \epsilon$, with ϵ small (but not too small). The $n = 1$ mode is expected to grow at a rate

$$\lambda = 1 - \frac{1}{1+\epsilon} \simeq \epsilon \tag{5.19}$$

This implies that the instability will develop on a slow timescale $T = \epsilon t$, which leads us to suppose that the weakly nonlinear solution is of the form $u(z, T)$. Furthermore, it is likely of small amplitude – the closer to onset we are, the weaker we expect the saturated amplitude of the perturbation to be. Without any more information, we can at least suppose that

$$u(z, T) = \epsilon^{\alpha} u_1(z, T) + \epsilon^{2\alpha} u_2(z, T) + \dots \tag{5.20}$$

The correct value of α is one of the outcomes of this problem that we actually wish to find.

Let's plug this expansion into the governing equation. We get

$$\begin{aligned} &\epsilon^{\alpha+1}\frac{\partial u_1}{\partial T} + \epsilon^{2\alpha+1}\frac{\partial u_2}{\partial T} + \dots - (\epsilon^{\alpha} u_1 + \epsilon^{2\alpha} u_2 + \dots) + \frac{1}{6}(\epsilon^{\alpha} u_1 + \epsilon^{2\alpha} u_2 + \dots)^3 \\ &= \frac{1}{1+\epsilon}\left(\epsilon^{\alpha}\frac{\partial^2 u_1}{\partial z^2} + \epsilon^{2\alpha}\frac{\partial^2 u_2}{\partial z^2} + \dots\right) \end{aligned} \tag{5.21}$$

To the lowest order, regardless of the value of α, we clearly have

$$\frac{\partial^2 u_1}{\partial z^2} + u_1 = 0 \tag{5.22}$$

which has the solution

$$u_1(z, T) = A_1(T)\sin(z) \tag{5.23}$$

Note that the sine solution was chosen here to satisfy the boundary conditions. Also note, in view of using the results of the previous section, that we have found a *non-zero* solution to the equation $\mathcal{L}u_1 = 0$, where $\mathcal{L} = d^2/dz^2 + 1$. It's trivial to show that this operator is self-adjoint with weight function $r(x) = 1$.

We now subtract this zeroth-order system, and divide the governing equation by ϵ^{α}. Since we do not know a priori what α is, we have to be careful in which order of ϵ to keep. For instance, we do not know if $\alpha \geq 1$ or $\alpha \leq 1$, or, for that matter, whether $2\alpha \geq 1$ or $2\alpha \leq 1$, and so forth. However, we do know that $2\alpha > \alpha$, and that $\alpha + 1 > \alpha$. For this reason, as a first pass at the problem, we keep all orders up to $\alpha + 1$ *and* 2α. We are then left with

$$\begin{aligned} &\epsilon\frac{\partial u_1}{\partial T} + \epsilon^{\alpha+1}\frac{\partial u_2}{\partial T} - \epsilon^{\alpha} u_2 - \epsilon^{2\alpha} u_3 + \frac{1}{6}\epsilon^{2\alpha} u_1^3 \\ &= -\epsilon\frac{\partial^2 u_1}{\partial z^2} + \epsilon^{\alpha}\frac{\partial^2 u_2}{\partial z^2} - \epsilon^{\alpha+1}\frac{\partial^2 u_2}{\partial z^2} + \epsilon^{2\alpha}\frac{\partial^2 u_3}{\partial z^2} \end{aligned} \tag{5.24}$$

Let's begin by being very naive, and assume $\alpha = 1$. Then we have, to the lowest order,

$$\frac{\partial u_1}{\partial T} - u_2 = -\frac{\partial^2 u_1}{\partial z^2} + \frac{\partial^2 u_2}{\partial z^2} \tag{5.25}$$

which implies that

$$\frac{\partial^2 u_2}{\partial z^2} + u_2 = \frac{\partial u_1}{\partial T} - u_1 = \left(\frac{\partial A_1}{\partial T} - A_1\right)\sin z \tag{5.26}$$

This equation for u_2 is no longer homogeneous, but still satisfies the same homogeneous boundary conditions as u_1 did. By Fredholm's alternative, because there is a non-zero solution u_1 to the homogeneous problem, we know that (5.97) only has solutions *if the solvability condition*

$$\langle\left(\frac{\partial A_1}{\partial T} - A_1\right)\sin z, u_1(z)\rangle = 0 \tag{5.27}$$

is satisfied. Given that $u_1(z) \propto \sin(z)$, and that $\langle \sin z, \sin z\rangle \neq 0$, this is equivalent to requiring

$$\frac{\partial A_1}{\partial T} - A_1 = 0 \tag{5.28}$$

which means that A_1 grows exponentially *without* saturating! This effectively recovers linear theory – it is not incorrect, but it is also not very helpful. It merely tells us that as long as the amplitude of the perturbation is smaller or of the other of ϵ, then it continues to grow exponentially. We were clearly a little bit too naive in our choice for α.

How can we account for nonlinear saturation? The key is to go back to equation (5.95), and more carefully inspect the various terms that arise. Since we want the nonlinear terms to come in at the same order as the $\partial u_1/\partial T$ term, we *have to have* $\epsilon^{2\alpha} = \epsilon$ – it's that simple. This means that $\alpha = 1/2$. Going back to equation (5.95) and keeping only the lowest-order terms in ϵ, we then get

$$\epsilon^{1/2}\frac{\partial^2 u_2}{\partial z^2} = -\epsilon^{1/2} u_2 \tag{5.29}$$

which implies, as before, that $u_2(z,T) = A_2(T)\sin(z)$. So far, that is not very useful. However, to the next order (that is, to order ϵ), we get

$$\frac{\partial u_1}{\partial T} - u_3 + \frac{1}{6}u_1^3 = -\frac{\partial^2 u_1}{\partial z^2} + \frac{\partial^2 u_3}{\partial z^2} \tag{5.30}$$

which implies

$$\begin{aligned}\frac{\partial^2 u_3}{\partial z^2} + u_3 &= \frac{\partial A_1}{\partial T}\sin z + \frac{1}{6}A_1^3\sin^3 z - A_1 \sin z \\ &= \left(\frac{\partial A_1}{\partial T} + \frac{1}{8}A_1^3 - A_1\right)\sin z - \frac{1}{24}A_1^3\sin(3z) \end{aligned} \tag{5.31}$$

using the identity $\sin^3(z) = (3/4)\sin(z) - (1/4)\sin(3z)$. By Fredholm's alternative, we know that this equation only has solutions if the dot product of $\sin z$ with its RHS is 0. While the $\sin(3z)$ term is indeed orthogonal to $\sin(z)$, the $\sin(z)$ term is not. We therefore need to have, as a solvability condition, that

$$\frac{\partial A_1}{\partial T} = A_1 - \frac{A_1^3}{8} \tag{5.32}$$

This equation, by contrast with the one we obtained for $\alpha = 1$, contains nonlinear terms that act to saturate the linear instability. In fact, we find that this ODE has the general solution

$$A_1(T) = \pm \frac{2\sqrt{2}e^T}{\sqrt{K + e^{2T}}} \tag{5.33}$$

where K is an integration constant that depends on the initial conditions. Since $u(z) = \epsilon^{1/2} A_1(T)\sin(z)$, with $\epsilon = R - 1$, then an approximation solution to (5.86) for R close to one is given by

$$u(z, T) = \pm \frac{2\sqrt{2(R-1)}e^T}{\sqrt{K + e^{2T}}} \sin(z) \tag{5.34}$$

As expected, for small T, u grows exponentially with growth rate 1, but as $T \to \infty$, $u \to 2\sqrt{2(R-1)}$. Note that the limit can be obtained without actually solving the equation for A_1, simply by requiring that the system be in a steady state. We find that, at steady state, $A_1^3 = 8A_1$, which either means that $A_1 = 0$ or that $A_1 = \pm 2\sqrt{2}$.

Hence three steady states exist. The two non-zero steady states are stable for $R > 1$, and have an amplitude that scales like the square root of the distance to the bifurcation point, while the null steady state is stable for $R < 1$. This is called a *Pitchfork bifurcation.* Figure 5.3 shows the ultimate steady states of $A_1(T)$ as a function of R.

A very important result of this analysis is that, by contrast with the example given in Section 5.1.3, we found that the nonlinear terms involved in the saturation of the linear instability were cubic, instead of quadratic. This, in fact, could have been predicted *simply by considerations of symmetry!*. Indeed, inspection of (5.86) shows that it is symmetric in z: if $u(z)$ is a solution of the equation, then $u(-z)$ is one too. Since $u_1(z, T) = A_1(T)\sin z$, then $u_1(-z, T) = -A_1(T)\sin z$ must also be a solution. This then implies that in the amplitude equation for A_1, if A_1 is a solution, then $-A_1$ must also be a solution. In other words, the amplitude equation must be invariant if $A_1 \to -A_1$. This would not have been satisfied had there been a quadratic term – in fact, had there been any even powers of A_1. Only odd powers of A_1 are allowed, simply because of the underlying symmetry $z \to -z$! These considerations are very powerful, and enable us to guess what the form of the amplitude equation of complicated systems may be *without* actually performing any asymptotic expansion. The only thing missing, in that case, is the value of the coefficients of each term (e.g. in this

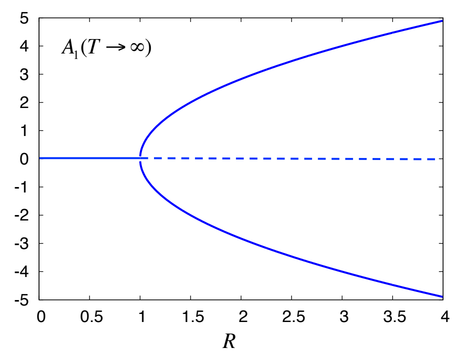

Figure 5.1: The steady states of $A_1(T)$ as a function of the input parameter R, illustrating the existence of a Pitchfork bifurcation. The solid lines mark stable steady states, while the dotted line mark the unstable steady state.

case, knowing that there must be a cubic term does not tell us its sign, nor the fact that the coefficient in front is 8). But knowing the power of the respective terms in the amplitude equation can already be enough to tell us what kind of bifurcation (pitchfork, saddle-node, transcritical, etc...) we might expect.

5.3.4 Weakly nonlinear theory for steady-state Rayleigh-Benard convection

Armed with a grand total of one previous attempt at deriving weakly nonlinear equations, we now attack the much harder problem of Rayleigh-Bénard convection.

Recall that the behavior of Rayleigh-Bénard convection for Ra just above onset is a rapid transition to another steady state that has 2D convective rolls. In what follows, we will therefore look for steady-state weakly nonlinear solutions of the problem. Despite being a rather different problem to the one we studied above, the method used is fairly similar: define a small parameter that is the distance to onset, expand all quantities in powers of that small parameter, and find solutions order-by-order, using the solvability condition.

Let's begin by re-writing the 2D governing equations in terms of the stream-function ϕ and the vorticity $\boldsymbol{\omega} = \omega \boldsymbol{e}_y$. First, recall that

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \nabla \times \nabla \times (\phi \boldsymbol{e}_y) = -\nabla^2 \phi \boldsymbol{e}_y \tag{5.35}$$

We can therefore define a scalar vorticity ω to be the component of vorticity in the y-direction. It is related to the streamfunction as

$$\omega = -\nabla^2 \phi \tag{5.36}$$

By taking the curl of the momentum equation, and extracting the y component of the resulting vorticity equation (which is the only component left in

2D), we get

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \omega = \mathrm{RaPr}\frac{\partial T}{\partial x} + \mathrm{Pr}\nabla^2 \omega \tag{5.37}$$

Using the stream function, the nonlinear term can be re-written as

$$\boldsymbol{u} \cdot \nabla \omega = \frac{\partial \phi}{\partial z}\frac{\partial \omega}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial \omega}{\partial z} = J(\omega, \phi) \tag{5.38}$$

where J is the Jacobian operator. Similarly, the temperature equation can be rewritten as

$$\frac{\partial T}{\partial t} + J(T, \phi) + \frac{\partial \phi}{\partial x} = \nabla^2 T \tag{5.39}$$

We now consider an expansion of these equation around marginal stability (that is, around Ra_c). To do so, we let

$$\mathrm{Ra} = (1 + \epsilon)\mathrm{Ra_c} \tag{5.40}$$

which defines ϵ. Here we assume that $\epsilon > 0$, since we have proved that there are no non-zero steady states for $\mathrm{Ra} < \mathrm{Ra}_c$. We then expand ϕ, ω and T as powers of ϵ^α, with α to be determined:

$$\begin{aligned} \phi(x, z) &= \epsilon^\alpha \phi_1(x, z) + \epsilon^{2\alpha}\phi_2(x, z) + \dots \\ \omega(x, z) &= \epsilon^\alpha \omega_1(x, z) + \epsilon^{2\alpha}\omega_2(x, z) + \dots \\ T(x, z) &= \epsilon^\alpha T_1(x, z) + \epsilon^{2\alpha} T_2(x, z) + \dots \end{aligned} \tag{5.41}$$

Note that it is not obvious a priori that the expansion should be like this: it could be that a different α may be necessary for each variable, or that the starting power may be different, etc... However, we will only know whether we are on the right track or not by plugging these expansions in the governing equations and finding out whether they give the right behavior.

We can already see that, indeed, the expansion has to be similar for ω and ϕ. We get, order by order,

$$\omega_i = -\nabla^2 \phi_i \tag{5.42}$$

For the other equations, at the lowest orders in ϵ^α, we get

$$\begin{aligned} &\epsilon^{2\alpha} J(\omega_1, \phi_1) + \dots = \mathrm{Ra_c Pr}(1 + \epsilon)\left(\epsilon^\alpha \frac{\partial T_1}{\partial x} + \epsilon^{2\alpha}\frac{\partial T_2}{\partial x}\right) + \mathrm{Pr}\nabla^2(\epsilon^\alpha \omega_1 + \epsilon^{2\alpha}\omega_2) + \dots \\ &\epsilon^{2\alpha} J(T_1, \phi_1) + \epsilon^\alpha \frac{\partial \phi_1}{\partial x} + \epsilon^{2\alpha}\frac{\partial \phi_2}{\partial x} \dots = \nabla^2(\epsilon^\alpha T_1 + \epsilon^{2\alpha} T_2) + \dots \end{aligned} \tag{5.43}$$

Dividing both equations by ϵ^α, we get

$$\begin{aligned} &\epsilon^\alpha J(\omega_1, \phi_1) + \dots = \mathrm{Ra_c Pr}(1 + \epsilon)\left(\frac{\partial T_1}{\partial x} + \epsilon^\alpha \frac{\partial T_2}{\partial x}\right) + \mathrm{Pr}\nabla^2(\omega_1 + \epsilon^\alpha \omega_2) + \dots \\ &\epsilon^\alpha J(T_1, \phi_1) + \frac{\partial \phi_1}{\partial x} + \epsilon^\alpha \frac{\partial \phi_2}{\partial x} \dots = \nabla^2(T_1 + \epsilon^\alpha T_2) + \dots \end{aligned} \tag{5.44}$$

We recover the linear equations at the zeroth order in ϵ:

$$\begin{aligned} \mathrm{Ra_c Pr}\frac{\partial T_1}{\partial x} - \mathrm{Pr}\nabla^4\phi_1 = 0 \\ -\frac{\partial \phi_1}{\partial x} + \nabla^2 T_1 = 0 \end{aligned} \quad (5.45)$$

using $\omega_1 = -\nabla^2\phi_1$. Of course, there is a trivial solution to these equations ($T_1 = \phi_1 = 0$). But we actually already know another non-trivial solution as well: at exactly $\mathrm{Ra} = \mathrm{Ra}_c$, Rayleigh-Bénard convection is just becoming linearly unstable, so there is one non-trivial mode with zero growth rate. We found it in Section 5.2.3, and it is:

$$\phi_1(x,z) = \hat{\phi}\sin(\pi z)\cos(k_c x) \quad (5.46)$$

where $k_c = \pi^2/2$, and where we have picked, arbitrarily, the cosine mode in the x direction. That mode is a solution to (5.116), with ω_1 and T_1 given by

$$\begin{aligned} \omega_1(x,z) &= \hat{\phi}(\pi^2 + k_c^2)\sin(\pi z)\cos(k_c x) \\ T_1(x,z) &= \hat{\phi}\frac{k_c \sin(\pi z)\sin(k_c x)}{\pi^2 + k_c^2} \end{aligned} \quad (5.47)$$

The only remaining unknown is the mode amplitude $\hat{\phi}$. Unfortunately, we can't get it from linear equations – this is also going to be one of the outcomes of the full weakly nonlinear problem.

At the next order, we would like to see the nonlinearities appear. This should, as in the toy problem studied earlier, give us an idea of what α should be. Here, a little bit of forward thinking goes a very long way. Note how the first order at which the nonlinearities *seem to* appear in the LHS of the momentum equation is $\epsilon^\alpha J(\omega_1, \phi_1)$, which we would very much like to be of the same order as the $\epsilon \mathrm{Ra_c Pr}\partial T_1/\partial x$ term in the RHS. This would require $\alpha = 1$. However, *doing that would be a mistake.* Indeed,

$$J(\omega_1, \phi_1) = -J(\nabla^2\phi_1, \phi_1) = -(\pi^2 + k_c^2)J(\phi_1, \phi_1) = 0 \quad (5.48)$$

and so the nonlinear terms only appear *at the next order*, and look like

$$\epsilon^{2\alpha}J(\omega_2, \phi_1) + \epsilon^{2\alpha}J(\omega_1, \phi_2) \quad (5.49)$$

Of course, it could be that saturation is caused by the nonlinear terms in the thermal energy equation (which is non-zero). After subtracting the lowest-order terms in equation (5.115) (which we have already equated to one another), and keeping only terms of order ϵ^α this time, we get

$$-\frac{\partial \phi_2}{\partial x} + \nabla^2 T_2 = J(T_1, \phi_1) \quad (5.50)$$

However, this doesn't tell us what α should be – and that is a strong hint that there is a serious problem in following this path (See the Appendix to find out what would have happened with $\alpha = 1$).

Going back to the momentum equation, if we actually want the nonlinear terms to balance $\epsilon\mathrm{Ra}_c\mathrm{Pr}\partial T_1/\partial x$, we are then forced to choose α such that $2\alpha = 1$, in other words, $\alpha = 1/2$. Using this in (5.115), and keeping only the leading order terms, we get to order $\epsilon^{1/2}$,

$$\begin{aligned} -\mathrm{Pr}\nabla^4\phi_2 + \mathrm{Pr}\mathrm{Ra}_\mathrm{c}\frac{\partial T_2}{\partial x} &= 0 \\ -\frac{\partial \phi_2}{\partial x} + \nabla^2 T_2 &= J(T_1, \phi_1) \end{aligned} \tag{5.51}$$

where (5.113) was used to eliminate ω_2.

At this point, we are getting close to a problem we have already seen: that of a forced linear equation, whose homogeneous counterpart has a non-trivial solution. We would therefore like to use Fredholm's alternative to get a solvability condition (which would then give us a condition on ϕ_1 and hence on $\hat{\phi}$). In order to do that, we have to prove that the linear problem is self-adjoint. As it turns out, this is not too difficult. Consider first that we now have a multidimensional operator $\mathcal{L}$, acting on the 2D vector of 2 functions $\boldsymbol{\phi} = (\phi, T)$. The linear problem can be recast as

$$\mathcal{L}\boldsymbol{\phi} = \begin{pmatrix} -\mathrm{Pr}\nabla^4 & \mathrm{Ra}_\mathrm{c}\mathrm{Pr}\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & \nabla^2 \end{pmatrix} = \begin{pmatrix} \phi \\ T \end{pmatrix} \tag{5.52}$$

We have already shown, for instance that

$$\mathcal{L}\boldsymbol{\phi}_1 = 0 \tag{5.53}$$

To show that $\mathcal{L}$ is self-adjoint, we to find an inner product for which $\langle \boldsymbol{\phi}_1, \mathcal{L}\boldsymbol{\phi}_2\rangle = \langle \mathcal{L}\boldsymbol{\phi}_1, \boldsymbol{\phi}_2\rangle$ for any two functions-vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ (i.e. not necessarily the ones associated with the expansion in ϵ). Trial and error shows that it is given by

$$\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_2\rangle = \int (\phi_1\phi_2 + \mathrm{Ra}_\mathrm{c}\mathrm{Pr}T_1T_2)dxdz \tag{5.54}$$

where the integral is taken over the domain D spanning z in $[0, 1]$, and assuming periodicity in x. Indeed, with successive integrations by parts (making sure that all of the boundary terms vanish), we have

$$\begin{aligned} & \langle \boldsymbol{\phi}_1, \mathcal{L}\boldsymbol{\phi}_2\rangle \\ = & \int \left[\phi_1\left(-\mathrm{Pr}\nabla^4\phi_2 + \mathrm{Ra}_\mathrm{c}\mathrm{Pr}\frac{\partial T_2}{\partial x}\right) + \mathrm{Ra}_\mathrm{c}\mathrm{Pr}T_1\left(-\frac{\partial\phi_2}{\partial x} + \nabla^2 T_2\right)\right] dxdz \\ = & \int \left[-\mathrm{Pr}\nabla^2\phi_1\nabla^2\phi_2 - \mathrm{Ra}_\mathrm{c}\mathrm{Pr}T_2\frac{\partial\phi_1}{\partial x} + \mathrm{Ra}_\mathrm{c}\mathrm{Pr}\phi_2\frac{\partial T_1}{\partial x} - \mathrm{Ra}_\mathrm{c}\mathrm{Pr}\nabla T_1\cdot\nabla T_2\right] dxdz \\ = & \int \left[\phi_2\left(-\mathrm{Pr}\nabla^4\phi_1 + \mathrm{Ra}_\mathrm{c}\mathrm{Pr}\frac{\partial T_1}{\partial x}\right) + T_2\left(-\mathrm{Ra}_\mathrm{c}\mathrm{Pr}\frac{\partial\phi_1}{\partial x} + \mathrm{Ra}_\mathrm{c}\mathrm{Pr}\nabla^2 T_1\right)\right] dxdz \\ = & \langle \boldsymbol{\phi}_2, \mathcal{L}\boldsymbol{\phi}_1\rangle \end{aligned} \tag{5.55}$$

Given that $\mathcal{L}$ is self-adjoint, we now know that, in order for (5.122) to have a solution, it is necessary that the inner product of $\boldsymbol{\phi}_1$ (the non-zero solution of the homogeneous problem) with its RHS be zero:

$$< \begin{pmatrix} \phi_1 \\ T_1 \end{pmatrix}, \begin{pmatrix} 0 \\ J(T_1, \phi_1) \end{pmatrix} >= 0 \tag{5.56}$$

This implies

$$\mathrm{Ra_c Pr} \int T_1(x,z) J(T_1, \phi_1) dx dz = 0 \tag{5.57}$$

Since

$$J(T_1, \phi_1) = \hat{\phi}^2 \frac{k_c^2 \pi}{\pi^2 + k_c^2} \frac{\sin(2\pi z)}{2} \tag{5.58}$$

that is, only depends on z, while $T_1 \propto \sin(k_c x)$, then indeed, (5.128) is true. We can therefore expect that there is a solution to this equation. A quick inspection reveals that, because the RHS of (5.122) is independent of x, the solution is also independent of x. Indeed, let's try $T_2 = T_2(z)$ and $\phi_2 = \phi_2(z)$. We then have the conditions

$$\begin{aligned} -\mathrm{Pr} \frac{d^4 \phi_2}{dz^4} &= 0 \\ \frac{d^2 T_2}{dz^2} &= \hat{\phi}^2 \frac{k_c^2 \pi}{\pi^2 + k_c^2} \frac{\sin(2\pi z)}{2} \end{aligned} \tag{5.59}$$

The solution to the first, subject to homogeneous boundary conditions, is simply $\phi_2(z) = 0$ (so $\omega_2 = 0$ too). The solution to the second yields

$$T_2(z) = -\hat{\phi}^2 \frac{k_c^2}{\pi^2 + k_c^2} \frac{\sin(2\pi z)}{8\pi} \tag{5.60}$$

We can now attack the next order in ϵ. Unfortunately, because $\phi_2 = \omega_2 = 0$, the nonlinear terms in the momentum equation *still* do not appear. However, by some stroke of luck, this does not matter, as we shall see. Let's proceed. We get,

$$\begin{aligned} 0 = \mathrm{Ra_c Pr} \left(\frac{\partial T_1}{\partial x} + \frac{\partial T_3}{\partial x} \right) - \mathrm{Pr} \nabla^4 \phi_3 \\ J(T_2, \phi_1) + \frac{\partial \phi_3}{\partial x} = \nabla^2 T_3 \end{aligned} \tag{5.61}$$

which can be rewritten as

$$\mathcal{L} \boldsymbol{\phi}_3 = \begin{pmatrix} -\mathrm{Ra_c Pr} \frac{\partial T_1}{\partial x} \\ J(T_2, \phi_1) \end{pmatrix} \equiv \mathcal{N}_3 \tag{5.62}$$

As usual, this only has solutions if

$$\langle \boldsymbol{\phi}_1, \mathcal{N}_3 \rangle = 0 \tag{5.63}$$

Given that

$$J(T_2, \phi_1) = -\frac{\partial T_2}{\partial z}\frac{\partial \phi_1}{\partial x} = -\hat{\phi}^3 \frac{k_c^3}{\pi^2 + k_c^2} \frac{\cos(2\pi z)}{4} \sin(\pi z)\sin(k_c x) \tag{5.64}$$

then

$$\begin{aligned}
&\langle \boldsymbol{\phi}_1, \mathcal{N}_3 \rangle \\
&= -\mathrm{Ra_c Pr} \int \left[\phi_1 \left(\frac{\partial T_1}{\partial x} \right) + T_1 \left(\hat{\phi}^3 \frac{k_c^3}{\pi^2 + k_c^2} \frac{\cos(2\pi z)}{4} \sin(\pi z)\sin(k_c x) \right) \right] dxdz \\
&= -\hat{\phi}^2 \mathrm{Ra_c Pr} \frac{k_c^2}{\pi^2 + k_c^2} \int \left[\sin^2(\pi z)\cos^2(k_c x) + \hat{\phi}^2 \frac{k_c^2}{\pi^2 + k_c^2} \frac{\cos(2\pi z)}{4} \sin^2(\pi z)\sin^2(k_c x) \right] dxdz \\
&= -\hat{\phi}^2 \frac{\mathrm{Ra_c Pr}}{2} \frac{k_c^2}{\pi^2 + k_c^2} \frac{\pi}{k_c} \int_0^1 \left[\sin^2(\pi z) + \hat{\phi}^2 \frac{k_c^2}{\pi^2 + k_c^2} \frac{\cos(2\pi z)}{4} \sin^2(\pi z) \right] dz \\
&= -\hat{\phi}^2 \frac{\mathrm{Ra_c Pr}}{4} \frac{k_c^2}{\pi^2 + k_c^2} \left[1 - \frac{\hat{\phi}^2}{8} \frac{k_c^2}{\pi^2 + k_c^2} \right]
\end{aligned} \tag{5.65}$$

For this quantity to be zero, we have to have

$$\hat{\phi}^2 = 8 \frac{\pi^2 + k_c^2}{k_c^2} \tag{5.66}$$

This finally tells us what the amplitude of the convective roll is! The weakly nonlinear solution can then be written as:

$$\begin{aligned}
\phi(x, z) &= \pm 2\sqrt{2}\epsilon^{1/2} \sqrt{\frac{\pi^2 + k_c^2}{k_c^2}} \sin(\pi z)\cos(k_c x) + O(\epsilon^{3/2}) \\
T(x, z) &= \pm \epsilon^{1/2} \frac{2\sqrt{2}}{\sqrt{\pi^2 + k_c^2}} \sin(\pi z)\sin(k_c x) - \epsilon \frac{\sin(2\pi z)}{\pi} + O(\epsilon^{3/2})
\end{aligned} \tag{5.67}$$

where we recall that ϵ was defined such that

$$\mathrm{Ra} = (1 + \epsilon)\mathrm{Ra}_c \tag{5.68}$$

As before, we see that the weakly nonlinear solution scales like the square root of the distance to the bifurcation parameter Ra_c, with two symmetric solutions. This reveals the onset of convection to be associated with a pitchfork bifurcation again, as in the previous example. And as in the previous example, that could have been guessed simply based on the symmetries of the problem!

The steps we went through to get to the weakly nonlinear solution, however, tell us a lot about the physics behind the nonlinear saturation of convection, at least close to onset. We see that the nonlinear terms in the momentum equation never came in. Instead saturation proceeds by the modification of the background state by the convective rolls. Indeed, the rolls, to order one, induced an order 2 change in the mean temperature profile. We see that instead of a linear profile, the nonlinear mean profile at saturation is given by

$$\int (\bar{T}(z) + \tilde{T}(x,z))dx = -z - \epsilon \frac{\sin(2\pi z)}{\pi} \tag{5.69}$$

This new profile has a more uniform temperature near the mid-point of the domain, and sharper gradients near the boundary (see Figure 5.4).

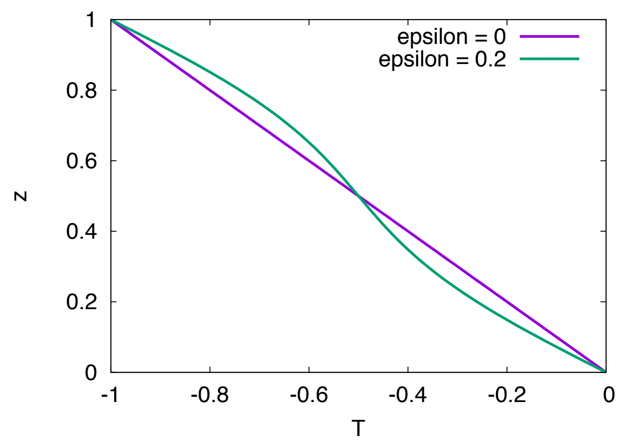

Figure 5.2: Mean temperature profile for the background state ($\epsilon = 0$) and for the weakly nonlinear steady state solution with $\epsilon = 0.2$.

Appendix

Suppose we had proceeded anyway, and required that α be equal to 1. The leading order equations then read

$$\begin{aligned} -\mathrm{Pr}\nabla^4\phi_2 + \mathrm{Ra_c Pr}\frac{\partial T_2}{\partial x} &= -\mathrm{Ra_c Pr}\frac{\partial T_1}{\partial x} \\ -\frac{\partial \phi_2}{\partial x} + \nabla^2 T_2 &= J(T_1, \phi_1) \end{aligned} \tag{5.70}$$

The associated solvability condition is

$$\int \left[\phi_1 \left(-\mathrm{Ra_c Pr}\frac{\partial T_1}{\partial x}\right) + \mathrm{Ra_c Pr} T_1 J(T_1, \phi_1)\right] dxdz = 0 \tag{5.71}$$

It can be shown with very little algebra that the only way of satisfying it is with $\hat{\phi} = 0$ (and therefore ϕ_1, T_1 and ω_1 are also 0). Arriving at this kind of result definitely means that our selection of expansion was incorrect.