Chapter 4

Dispersive waves

In this Chapter we will study a much broader class of waves called *dispersive* waves, for which the phase speed and group speeds are different. Since non-dispersive wave must satisfy $\omega = \alpha|k|$, dispersive waves include any wave whose dispersion relation is *not* of that form. Most fluid waves are in fact dispersive. We will also see that the equations governing these dispersive waves can look very different from what we commonly think of as "the wave equation" (e.g. variants of $\partial_{tt} f = c^2 \nabla^2 f$). This can sometimes give rise to more existential questions on the lines of *What is a wave?* (see the excellent book by Whitham for a discussion of this simple and amazingly non-trivial question). Interestingly, we will see that while some of the properties of dispersive waves are very different from those of non-dispersive ones, the tools used to study them are essentially the same as those introduced in Chapter 2. To see this, we will study two very standard, and yet very different kinds of waves: internal gravity waves, and surface waves (including surface gravity waves and capillary waves).

4.1 Internal gravity waves

4.1.1 General ideas

Internal gravity waves are waves whose restoring force is buoyancy, that is, Archimede's force. If a fluid element has density ρ_e, and is immersed in a background fluid of density ρ_b, the force exerted on the element is

$$\mathbf{F}_b = (\rho_e - \rho_b)\boldsymbol{g} \tag{4.1}$$

If the element is denser than the background, then the force is pointing in the direction of gravity (it sinks), and if it is less dense, then the force is in the direction opposite to gravity (it rises). A nice way of testing the buoyancy force is to breathe deeply and exhale deeply while floating in a swimming-pool. The more air in our lungs, the less dense we are overall, and the higher we float. The less air in the lungs, the more we sink.

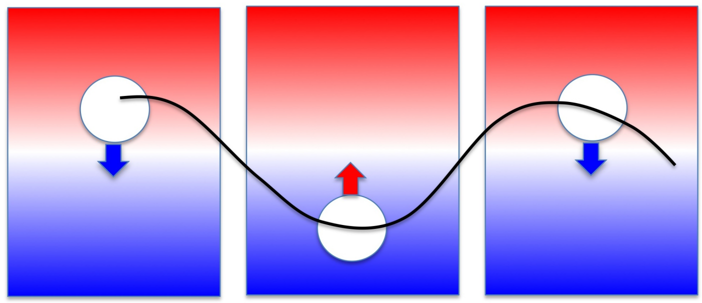

Figure 4.1: Schematic example of a gravity wave whose associated fluid motion is purely vertical.

Consider now a stratified fluid, whose density decreases with height (see Figure 4.1). This commonly occurs for example in the tropical ocean, where the warmer water in the surface layers heated by the Sun is less dense than the cold bottom water. If a fluid element is raised from its original rest position, it finds itself in a new background that is less dense than itself, and begins to sink back. It accelerates thanks to the buoyancy force until it reaches its original position again. When this happens, the buoyancy force drops to zero but the element is still moving downward, where it gradually finds itself lighter than the environment. The upward buoyancy force decelerates it, until it finally stops moving down, and begins to move up again. The process repeats in a wave-like fashion.

The oscillatory motion resulting from the original displacement is a called a gravity wave[1]. Gravity waves are very common in stably stratified media. They occur in the deep ocean, in the atmosphere, in the interior of stars, etc... Gravity waves on Earth can reach huge amplitudes. For instance, some gravity waves in the Ocean, caused by the tides and amplified by the bottom topography, can reach amplitudes of several hundred meters (!). Interestingly, they are barely visible at the surface, but a fluid element (or any animal, submarine, etc.) finding itself in the wave can be lifted up and down by this amount. Similarly enormous waves can be detected in the lee-side wind-wake of tall mountain ridges, and commonly trigger the formation of lenticular clouds or other fascinating cloud patterns. In stellar interiors, gravity waves can be excited by tides caused by a companion star or large planet, or by plunging convective plumes hitting a stably stratified radiative region.

4.1.2 Internal waves in an infinite domain

The simplest model for internal gravity waves consists of an infinite stratified domain, and uses the Boussinesq approximation (making all the necessary rel-

[1]Not to be confused with gravitational waves in astrophysics.

evant assumptions). This involves assuming the existence of a background in hydrostatic and thermal equilibrium. In all that follows, we use the same notations as in Chapter 3. If we further assume that the waves are adiabatic and non-viscous, the linearized momentum equation, thermal energy equation and equation of state are

$$\begin{aligned} \rho_m \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} &= -\nabla \tilde{p} - \tilde{\rho} g \boldsymbol{e}_z \\ \frac{\partial \tilde{T}}{\partial t} + \tilde{w} \Theta_{0z} &= 0 \\ \frac{\tilde{\rho}}{\rho_m} &= -\alpha \tilde{T} \end{aligned} \tag{4.2}$$

where Θ_{0z} (which has to be positive to ensure the fluid is stably stratified) is either the background temperature gradient, or the difference between the background temperature gradient and the adiabatic temperature gradient depending on whether we are considering waves in a liquid or a gas. Note how we have neglected all nonlinear terms assuming that the perturbations to the background (marked with tildes) are small. In what follows, we now drop the tildes for convenience of notation. While α and g are usually thought to be constant (at least on Earth, which is where this application takes us), we let Θ_{0z} be a slowly varying function of position and time. For simplicity, we only consider here 2D waves – the main results carry over in 3D, but in 2D the mathematical derivation is considerably more elegant because we can use a *streamfunction*.

Assuming that the flow is only in the (x, z) plane, and independent of y, we define the streamfunction ϕ as the scalar function that satisfies

$$\boldsymbol{u} = -\nabla \times (\phi \boldsymbol{e}_y) \rightarrow u = \frac{\partial \phi}{\partial z} \text{ and } w = -\frac{\partial \phi}{\partial x} \tag{4.3}$$

With this definition, the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is automatically satisfied, and the curl of the momentum equation becomes a scalar equation:

$$\frac{\partial}{\partial t}\left(\nabla^2 \phi\right) = -\alpha g \frac{\partial T}{\partial x} \tag{4.4}$$

Meanwhile, the thermal energy equation becomes

$$\frac{\partial T}{\partial t} = \Theta_{0z} \frac{\partial \phi}{\partial x} \tag{4.5}$$

so that, taking the time derivative of the momentum equation we get

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 \phi\right) = -\frac{\partial}{\partial x}\left(\alpha g \Theta_{0z} \frac{\partial \phi}{\partial x}\right) \tag{4.6}$$

The quantity $\sqrt{\alpha g \Theta_{0z}}$ is usually called the *buoyancy frequency* and sometimes the *Brunt-Väisälä frequency*, and denoted by the letter N. We can then finally write

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 \phi\right) = -\frac{\partial}{\partial x}\left(N^2 \frac{\partial \phi}{\partial x}\right) \tag{4.7}$$

where N could be a slowly varying function of position and time. This is the governing equation for internal gravity waves. As we can see, it doesn't really look much like the standard wave equation.

Finally, note that, as in the case of pressure waves, we can also create an energy conservation equation. To do so, we dot the momentum equation with $\boldsymbol{u}$ again, to get

$$\frac{\rho_m}{2}\frac{\partial|\boldsymbol{u}|^2}{\partial t} = -\boldsymbol{u}\cdot\nabla p + \rho_m \alpha g w T \tag{4.8}$$

To get an equation for wT, we multiply the temperature equation by T, which yields

$$\frac{1}{2}\frac{\partial T^2}{\partial t} + wT\Theta_{0z} = 0 \tag{4.9}$$

so that, at all times, we have

$$\frac{\rho_m}{2}\frac{\partial|\boldsymbol{u}|^2}{\partial t} = -\boldsymbol{u}\cdot\nabla p - \frac{\rho_m\alpha^2 g^2}{2N^2}\frac{\partial T^2}{\partial t} \tag{4.10}$$

We can recast this, as in the case of pressure waves, in a more conservative form by integrating over a fixed domain D, and assuming either periodic boundary conditions or that all perturbations vanish on the surface of D. Then

$$\frac{\partial E_D}{\partial t} = \frac{\rho_m\alpha^2 g^2 T^2}{2}\frac{\partial}{\partial t}\left(\frac{1}{N^2}\right) \tag{4.11}$$

where

$$E_D = \rho_m \int_D \left(\frac{|\boldsymbol{u}|^2}{2} + \frac{\alpha^2 g^2}{2}\frac{T^2}{N^2}\right) dV \tag{4.12}$$

As in the case of pressure waves, we find that the total energy in this domain (with these boundary conditions) only changes if the background, characterized by N, depends on time.

4.1.3 Monochromatic plane wave solutions

The dispersion relation for internal gravity waves

Let's first consider the case where N is constant. Since the governing wave equation (4.7) has constant coefficients, we can look for plane monochromatic wave solutions of the form

$$\phi(x,z,t) = \hat{\phi}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \tag{4.13}$$

Plugging this solution into (4.7) we find that

$$k^2\omega^2 = k_x^2 N^2 \tag{4.14}$$

where $k^2 = k_x^2 + k_y^2$, so the dispersion relation is simply

$$\omega^2 = \frac{k_x^2}{k^2}N^2 \tag{4.15}$$

This expression has a number of important consequences. First, note that by contrast with the case of pressure waves, the dispersion relation depends on the direction of the wave-vector. These waves are *anisotropic* (meaning that they know about a preferred direction). This is not surprising – since gravity is the restoring force, and since that force only operates in a particular direction, the waves *have* to know the difference between the vertical and horizontal directions.

Next, note that ω *only* depends on the direction of $\boldsymbol{k}$ and not on its magnitude (to see this, simply write $k_x/|k|$ as the cosine of the angle between $\boldsymbol{k}$ and the horizontal for instance). This means that ω is independent of the wavelength of the wave – quite different from sound waves where ω is proportional to $|k|$.

Finally, note that ω cannot be larger than N, or in other words, the maximum frequency of an internal gravity wave is N. This is achieved for waves for which $k_x^2 = k^2$, which have $k_z = 0$. These are modes that only oscillate up and down without horizontal motion. As the wavevector goes from purely vertical to more-and-more horizontal (i.e. as the amplitude of k_z grows), ω decreases.

Local properties of the waves, and the phase speed

Let's first think about what the wave-crests look like. As usual, they are given by the maximas and minimas of the phase function $\theta = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$, and are simply parallel straight lines. Furthermore, since the gradient of θ is $\boldsymbol{k}$ (by definition) and since the gradient of a function is *perpendicular* to its isolines, then we know that $\boldsymbol{k}$ must be perpendicular to the wave-crests. So far, this is just as in the case of monochromatic plane pressure waves.

We now consider the fluid motion associated with these waves. Since $\boldsymbol{u} = -\nabla \times (\phi \boldsymbol{e}_y)$, we can rewrite this as $\boldsymbol{u} = -i\phi \boldsymbol{k} \times \boldsymbol{e}_y$. We see from this expression that the fluid moves *perpendicular* to the wave-vector $\boldsymbol{k}$, and therefore *parallel* to the wave crests. In this sense, gravity waves are *transverse* waves, by contrast with the pressure waves which are *longitudinal* waves (where the fluid velocity was parallel to $\boldsymbol{k}$). Another way of seeing this is to look at how a fluid element in this flow actually moves. The equations controlling the position of the fluid element $(x_e(t), z_e(t))$ are

$$\frac{dx_e}{dt} = u \text{ and } \frac{dz_e}{dt} = w \tag{4.16}$$

Recalling that only the real part of a solution has a physical meaning, so we have

$$\begin{aligned} \frac{dx_e}{dt} &= \Re(ik_z \hat{\phi} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}) \\ \frac{dz_e}{dt} &= -\Re(ik_x \hat{\phi} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}) \end{aligned} \tag{4.17}$$

Let's decide for instance that $\hat{\phi}$ is real (this sets the phase of the oscillation to be a sine or a cosine). We then have

$$\frac{dx_e}{dt} = -\hat{\phi} k_z \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$$
$$\frac{dz_e}{dt} = \hat{\phi} k_x \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \tag{4.18}$$

which can be integrated to give

$$x_e(t) = x_0 - \hat{\phi}\frac{k_z}{\omega}\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$$
$$z_e(t) = z_0 + \hat{\phi}\frac{k_x}{\omega}\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \tag{4.19}$$

Since the two are in phase, the particle paths are straight segments of horizontal length $L_x = 2\hat{\phi}k_z/\omega$ and vertical height $L_z = 2\hat{\phi}k_x/\omega = 2\hat{\phi}k/N$.

Finally, let's see how the wave crests move. To do this consider the function $k_x x + k_z z - \omega t$: at constant x, this function is invariant if $z-(\omega/k_z)t$ is constant, or in other words if z moves with velocity ω/k_z. At constant z, similarly, this function is invariant if x moves with velocity ω/k_x. This would suggest that the phase velocity is given by $(\omega/k_x, \omega/k_z)$. However, the phase velocity is not really a vector. If this were true, we should be able to write the phase as a function of $\boldsymbol{x} - \boldsymbol{c}_p t$, which is clearly not the case. In general, we then simply define the phase velocity to be

$$c_p = \frac{\omega}{|k|} = \frac{|k_x|}{k^2}N \tag{4.20}$$

for gravity waves.

Figure 4.2 summarizes what we have learned so far, and gives more insight into the physics of the waves. It shows snapshots of a monochromatic plane wave with $k_x = k_z$ at $t = 0$ and at a time $t = \delta t$ very shortly thereafter. At $t = 0$, focussing on the area near the center of the image we see the fluid going down and to the left, parallel to the wave crests, transporting heat from higher layers to lower ones. The initial temperature perturbation at $t = 0$ in that region is zero, but gradually increases as a result of the advected heat. At the slightly later time, we see that the temperature has increased, and the warmer fluid being somewhat more buoyant, begins to resist the downward fluid flow which slows down. As a result, the wave crest apparently propagates following the green arrows (from bottom right to top left). Later on (not shown) the temperature in the center region will continue to increase until the resulting buoyancy becomes sufficient to cause a reversal of the flow. When this happens, the fluid will start to move from the bottom left to the top right, bringing cold fluid up to gradually cool the center region - and the cycle will eventually repeat.

4.1.4 Superposition of waves and the group speed

Let's now consider the superposition of 2 plane monochromatic waves, as we did in the case of pressure waves, with wavenumbers that are close to one another.

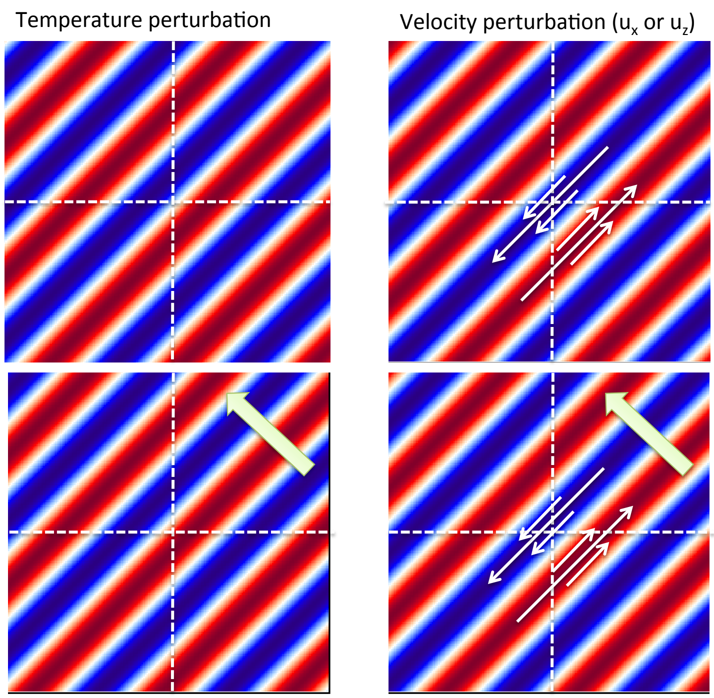

Figure 4.2: Left: Temperature perturbations at two different times ($t = 0$ and $t = \delta t$ where δt is small) for a plane monochromatic wave with $k_x = k_z$. Right: Velocity perturbations (both u_x or u_z, as the two are exactly in phase) at the same times. The dashed lines mark the center of the image, to guide the eye. The green arrows mark the direction of propagation of the wave crests.

We have

$$\begin{aligned} \phi(x,z,t) &= \hat{\phi}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega(\boldsymbol{k})t} + \hat{\phi}(\boldsymbol{k}+d\boldsymbol{k})e^{i(\boldsymbol{k}+d\boldsymbol{k})\cdot\boldsymbol{x}-i\omega(\boldsymbol{k}+d\boldsymbol{k})t} \\ &= \hat{\phi}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega(\boldsymbol{k})t}\left[1+\hat{\phi}(\boldsymbol{k})e^{i(d\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{c}_g t))} + O(|dk|)\right] \end{aligned} \quad (4.21)$$

where

$$\boldsymbol{c}_g = \left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_z}\right) = \left(\frac{N}{k} - \frac{Nk_x^2}{k^3}, -\frac{Nk_xk_z}{k^3}\right) = \frac{N}{k}\left(\boldsymbol{e}_x - k_x\frac{\boldsymbol{k}}{k^2}\right) \quad (4.22)$$

since

$$\omega(\boldsymbol{k}+d\boldsymbol{k}) = \omega(\boldsymbol{k}) + \frac{\partial\omega}{\partial k_x}dk_x + \frac{\partial\omega}{\partial k_y}dk_y + \ldots = \omega(\boldsymbol{k}) + \boldsymbol{c}_g\cdot d\boldsymbol{k} + \ldots \quad (4.23)$$

We therefore see that the superposition of these two waves with nearly the same wavenumber is, to a first approximation, equal to a rapidly oscillating function times a slowly varying one, as in the case of pressure waves. The slowly varying function has wave-crests that move with the velocity $\boldsymbol{c}_g$ defined above – since the phase of that function is invariant when $\boldsymbol{x} = \boldsymbol{c}_g t+$ a constant. This reveals $\boldsymbol{c}_g$ to be the *group* velocity, and this time, it is a well-defined vector. We now see, from this alternative derivation of the group velocity, why it is given by the gradient of ω with respect to $\boldsymbol{k}$.

Note that while we had $c_p = |c_g|$ for pressure waves, here the amplitudes of the phase and group speeds are quite different. We have

$$|\boldsymbol{c}_g| = c_g = \frac{N|k_z|}{k^2} \tag{4.24}$$

while c_p was given by $c_p = N|k_x|/k^2$. Unless $k_x = k_z$, the two velocities have different magnitudes. Furthermore,

$$\boldsymbol{c}_g \cdot \boldsymbol{k} = \frac{N}{k}\left(\boldsymbol{e}_x - k_x \frac{\boldsymbol{k}}{k^2}\right) \cdot \boldsymbol{k} = 0 \tag{4.25}$$

or, in other words, the group speed is perpendicular to the wave vector. Since the latter is itself perpendicular to the wave crests, then the group speed is *along* the wave crests (which is really not very intuitive!).

Finally, note that

$$\boldsymbol{c}_g \cdot \boldsymbol{e}_z = \frac{N}{k}\left(\boldsymbol{e}_x - k_x \frac{\boldsymbol{k}}{k^2}\right) \cdot \boldsymbol{e}_z = -N\frac{k_x k_z}{k^3} \tag{4.26}$$

while the phase velocity in the z direction is $\omega/k_z = Nk_x/(kk_z)$. This implies that the two are always of opposite sign. Hence if the group speed points up, the phase will appear to propagate downward, and vice versa. Since it is the group speed that carries any significant information (such as energy, for instance), and since the latter always points *away* from the source, we are forced to conclude that the wave crests in a gravity wave will always appear to be moving *towards* the source (which is even less intuitive)! These effects are all illustrated in Figure 4.3

4.1.5 Wave packet equations

As in the case of pressure waves, we can study the evolution of internal gravity waves in a medium with non-constant N by consider a wave packet solution. We construct it by considering the function ϕ and set

$$\phi(x, z, t) = A(X, Z, \tau)e^{i\theta} \tag{4.27}$$

where the slow time is now called τ to avoid any confusion with the temperature field T. As before, we have $\boldsymbol{k} = \nabla\theta$ and $\omega = -\partial\theta/\partial t$, which implies that

$$\frac{\partial \boldsymbol{k}}{\partial \tau} + \nabla_\epsilon \omega = 0 \tag{4.28}$$

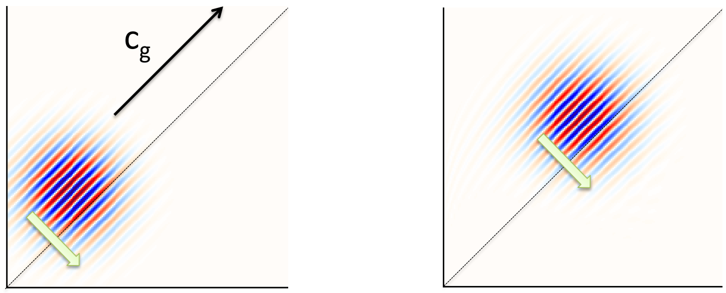

Figure 4.3: Temperature perturbations at two different times ($t = 0$ and $t = T$) for a wave packet, with $k_x = k_z$. The black arrow marks the direction of the group velocity. The green arrows mark the direction of propagation of the wave crests. Note the apparent distortion of the wave packet near the edges, which arises due to numerical artefacts introduced by the numerical integration scheme used.

The wave packet equations can be derived, step-by-step, as we did in the case of pressure waves, by plugging the ansatz (4.27) into (4.7), and equating the various orders in ϵ to one-another. This is left as a possible course project. However, the result can also be directly obtained *practically without any further math*, by recalling that

- The dispersion relation remains unchanged, so

$$\omega^2 = \frac{k_x^2}{k^2} N^2 \rightarrow \omega = \Omega(\boldsymbol{k}; \boldsymbol{X}, \tau) = \pm \frac{k_x}{k} N(\boldsymbol{X}, \tau) \tag{4.29}$$

  Note that the direction of the group velocity depends on the branch of solution selected for the dispersion relation.

- The evolution of the frequency is given by

$$\frac{\partial \omega}{\partial \tau} + \boldsymbol{c}_g \cdot \nabla_\epsilon \omega = \frac{\partial \Omega}{\partial \tau} = \pm \frac{k_x}{k} \frac{\partial N}{\partial \tau} \tag{4.30}$$

- The evolution equation for the wavevector is given by

$$\frac{\partial \boldsymbol{k}}{\partial \tau} + \boldsymbol{c}_g \cdot \nabla_\epsilon \boldsymbol{k} = -\nabla_\epsilon \Omega = \mp \frac{k_x}{k} \nabla_\epsilon N \tag{4.31}$$

- The evolution equation for the amplitude can be found by energy conservation arguments, where the energy conservation equation is given by

$$\frac{\partial E}{\partial \tau} + \nabla_\epsilon \cdot (\boldsymbol{c}_g E) = \text{RHS} \tag{4.32}$$

To use this last piece of information, we first have to determine what E is in terms of A, and then what the RHS from energy conservation in the original equations.

To find what E is in terms of A, note that (to the lowest order in ϵ)

$$\begin{aligned} u &= \partial\phi/\partial z = ik_z A e^{i\theta} \\ w &= -\partial\phi/\partial x = -ik_x A e^{i\theta} \\ T &= \frac{\Theta_{0z}k_x}{\omega} A e^{i\theta} \end{aligned} \tag{4.33}$$

where the last equation comes from $\partial T/\partial t + \Theta_{0z} w = 0$. Plugging these into E as obtained in equation (4.12), we find that the energy is

$$\begin{aligned} E &= \frac{\rho_m}{2}(|u|^2 + |w|^2) + \frac{\rho_m \alpha^2 g^2}{2}\frac{|T|^2}{N^2} \\ &= \frac{\rho_m}{2}(k_x^2 + k_z^2)|A|^2 + \frac{\rho_m \alpha^2 g^2 \Theta_{0z}^2}{2N^2}\frac{k_x^2}{\omega^2}|A|^2 \\ &= \rho_m k^2 |A|^2 \end{aligned} \tag{4.34}$$

Next, recall that the RHS of the energy equation (4.11) was

$$\text{RHS} = \frac{\rho_m \alpha^2 g^2 T^2}{2}\frac{\partial}{\partial t}\left(\frac{1}{N^2}\right) \tag{4.35}$$

Re-written in terms of A, we get

$$\text{RHS} = \frac{\rho_m \alpha^2 g^2}{2}\frac{\Theta_{0z}^2 k_x^2}{\omega^2}|A|^2 \frac{\partial}{\partial t}\left(\frac{1}{N^2}\right) = \frac{\rho_m N^2 k^2}{2}|A|^2 \frac{\partial}{\partial t}\left(\frac{1}{N^2}\right) \tag{4.36}$$

Combining all this information finally implies that the wave amplitude equation can be obtained from:

$$\frac{\partial}{\partial \tau}(|A|^2 k^2) + \nabla \cdot \left(\boldsymbol{c}_g k^2 |A|^2\right) = \frac{N^2 k^2 |A|^2}{2}\frac{\partial}{\partial \tau}\left(\frac{1}{N^2}\right) \tag{4.37}$$

4.1.6 Example of application: trapping of internal waves in the thermocline

The oceanic thermocline is a stable, strongly stratified region located between the surface mixed layer (which typically extends down to 50-100m, depending on the season) and the deep ocean (below about 200m, depending on the season). It is shallow enough to be heated by the surface illumination, but deep enough not to be mixed by the surface wind-induced turbulence. In both regions above and below the thermocline, the stratification is weak (N is close to 0), while N can be quite significant in the thermocline.

Because of this, internal waves generated in the thermocline are trapped. To see this, let's model N in the thermocline as a parabola, taking $Z = 0$ at the interface with the upper mixed layer, and $Z = H$ at the bottom of the thermocline (and beginning of the deep ocean water). We thus have

$$N(Z) = N_0 Z(H - Z) \tag{4.38}$$

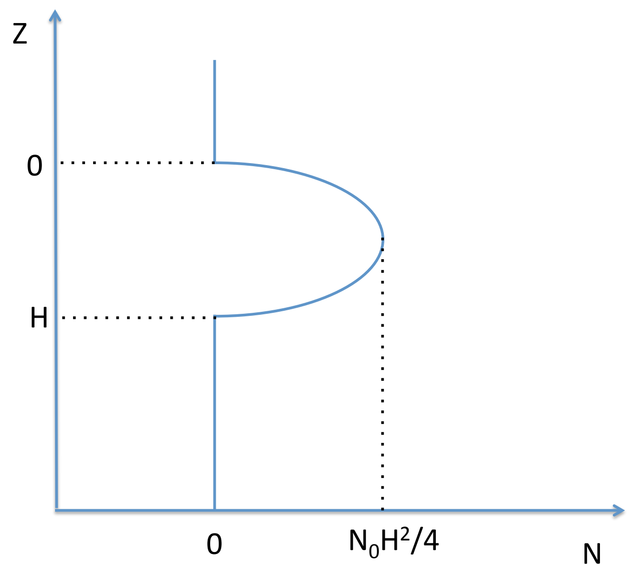

Figure 4.4: Schematic profile of the buoyancy frequency in the thermocline in this toy model.

The profile is shown in Figure 4.4

Using the ray tracing methods introduced in the case of pressure waves in stars, we see that since N only depends on Z, any internal wave generated in the thermocline will preserve its frequency and horizontal wavenumber. As a result, k_z must vary so that

$$\omega^2 = N^2 \frac{k_x^2}{k_x^2 + k_z^2} \rightarrow k_z = \pm \left(\frac{N(Z)^2}{\omega^2} - 1 \right)^{1/2} k_x \tag{4.39}$$

This implies that k_z tends to 0 whenever $N = \omega$ (at these points, we anticipate that the waves must refract, i.e. change direction). The ray paths of the waves can by found by solving the equation

$$\frac{dZ}{dX} = \frac{\boldsymbol{c}_g \cdot \boldsymbol{e}_z}{\boldsymbol{c}_g \cdot \boldsymbol{e}_x} = \frac{-\frac{N k_x k_z}{k^3}}{\frac{N}{k} - \frac{N k_x^2}{k^3}} = \frac{-k_x k_z}{k^2 - k_x^2} = -\frac{k_x}{k_z} = \left(\frac{N(Z)^2}{\omega^2} - 1 \right)^{-1/2} \tag{4.40}$$

Note that this does not appear to have simple analytical solutions, but a solution to this equation can easily be found by numerical integration (see Figure 4.5).

We see that the thermocline acts as a wave-guide for internal waves, which bounce back and forth between $Z_{\min}$ and $Z_{\max}$, the two depths where N is equal to the initial frequency of the wave. These upper and lower turning points are thus found by solving the equation $N(Z) = \omega$, which in our case reduces to

$$N_0 Z(H - Z) = \omega \tag{4.41}$$

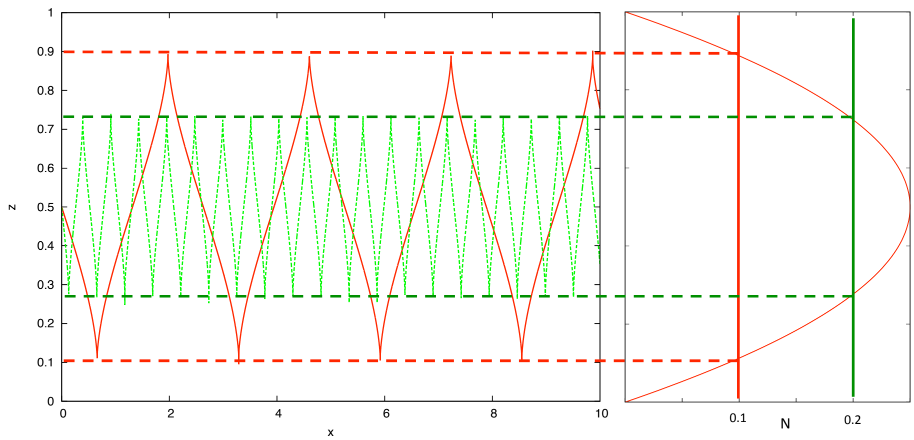

Figure 4.5: Trapping of gravity waves by the thermocline. The ray paths bounce between the positions where $N(Z) = \omega$ where ω is the frequency of the wave ($\omega = 0.1$ for the red path, and $\omega = 0.2$ for the green path. Here we have used $N_0 = H = 1$.

As in the case of pressure waves, we anticipate that the refraction induces a phase shift in the wave. This shift can be studied formally (cf. Project).

Finally, note that even in the absence of trapping (i.e. even when N is constant), internal waves usually cause very little motion near the surface. This will be studied in depth later. Internal waves of very large amplitudes (i.e. nonlinear waves) can be caused by tidal flows interacting over shallow topography, for instance, or other mechanisms such as submarines passing through the thermocline, deep large-scale eddies, etc. However, these waves are nearly invisible at the surface!

4.1.7 Generation of steady internal waves by flow over topography.

This section is adapted from Chapter 10 of the textbook "Waves in the Ocean and atmosphere" by J. Pedlosky. See also the book "Internal Gravity Waves" by B. Sutherland.

In this section, we change gear somewhat and move away from the wave packet approximation to look at the problem of internal waves generated in a fluid by the interaction of a steady flow with topography. This is a very common situation in the atmosphere, where internal waves are generated as winds pass over hills and mountains. It is also a very common scenario in the ocean, as large-scale azimuthal or meridional currents pass over bottom topography. In a number of cases, this interaction leads to the generation of steady wave patterns. A common, and visually stunning example is that involved in the formation of lenticular clouds near mountains. These are caused by the condensation of water due to the change in temperature/pressure associated with the waves. We will now see how these waves are generated, and how to model them.

A simple model

We begin by considering a minimal model of this problem. We model the ocean or atmosphere as a semi-infinite infinite domain, with the average height of the bottom boundary located at $z = 0$ and z unbounded in the positive direction. We consider again for simplicity that the flow is 2D. The domain is infinite in the x direction.

The lower boundary is not flat, and its height is given by the function $z = h(x)$. For simplicity, we will assume that $h(x)$ is periodic, and of the form

$$h(x) = h_0 \sin(kx) \tag{4.42}$$

Note that since the problem we are about to study is linear, a general solution for arbitrary topography can be obtained by linear combination of the solutions for periodic topography, in the Fourier sense.

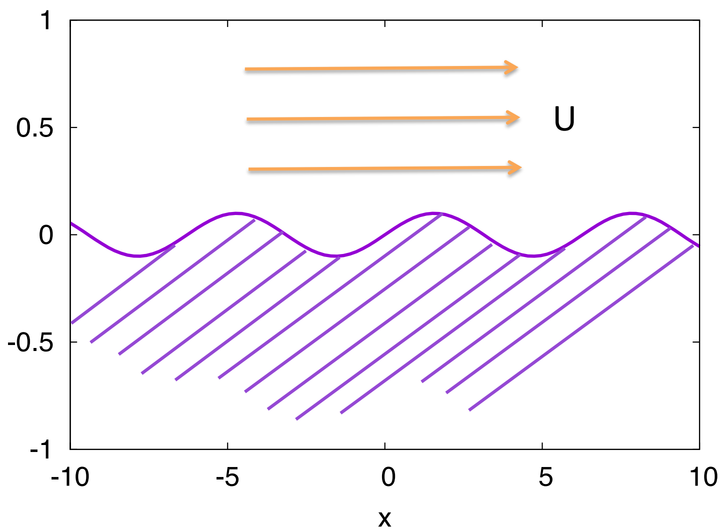

Figure 4.6: Schematic example of flow over sinusoidal topography, here with $h_0 = 0.1$ and $k = 1$.

Above $z = h(x)$ lies a stratified fluid, which we assume has a constant buoyancy frequency N. The fluid is not at rest, however, and supports a background horizontal flow $\boldsymbol{U} = U\boldsymbol{e}_x$, which is assumed to be constant for simplicity. That flow interacts with the bottom topography and drives the formation of internal gravity waves. We aim to characterize this internal wave field. In all that follows, we will assume that the amplitude of the waves generated is small, and does not affect the mean flow much in return. We will determine a posteriori under which conditions this may be true.

We assume that the total fluid velocity, including the background and the perturbations, is given by $\boldsymbol{u} = U\boldsymbol{e}_x + \tilde{\boldsymbol{u}}$ and satisfies the Boussinesq equations derived in Chapter 3. Because of the added presence of the background flow, the wave equation (4.7) derived earlier to model internal gravity waves is no longer valid. We can follow the same the steps however, taking the background flow into account, to find out how it should be modified for our current purpose.

With $\boldsymbol{u} = U\boldsymbol{e}_x + \tilde{\boldsymbol{u}}$, we have

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + U\frac{\partial \tilde{\boldsymbol{u}}}{\partial x} + \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} \simeq \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + U\frac{\partial \tilde{\boldsymbol{u}}}{\partial x} \tag{4.43}$$

in the linear approximation. A similar result applies for DT/Dt, so that the governing equations (4.2) remain the same, except that all the $\partial/\partial t$ must be replaced by $\partial/\partial t + U\partial/\partial x$. This then suggests that we should introduce a new variable $\xi = x - Ut$, for which $\partial/\partial x \to \partial/\partial \xi$ and $\partial/\partial t \to \partial/\partial t - U\partial/\partial \xi$. We then finally recover exactly the gravity wave equation, with x replaced by ξ. Physically speaking, this means that we have simply changed coordinates to put ourselves in a frame of reference that is moving with the mean flow. In that reference frame, there is no mean flow so (4.7) applies. The boundary conditions, however, become time-dependent.

To model the boundary conditions, note that the bottom boundary is impermeable to the fluid, which means that the component of the fluid velocity normal to the boundary must be 0. Since the normal to the surface $z = h(x)$ is given by $\nabla(z - h(x)) = \boldsymbol{e}_z - \frac{\partial h}{\partial x}\boldsymbol{e}_x$, the boundary condition reads

$$\boldsymbol{u} \cdot \nabla(z - h(x)) = 0 \to \boldsymbol{u} \cdot \boldsymbol{e}_z = \boldsymbol{u} \cdot \frac{\partial h}{\partial x}\boldsymbol{e}_x \tag{4.44}$$

Linearizing this (assuming $\tilde{u}$ is small compared with U), we have

$$\tilde{w} = U\frac{\partial h}{\partial x} = Uh_0 k\cos(kx) \tag{4.45}$$

We now see that the condition $|\tilde{\boldsymbol{u}}| \ll U$ is equivalent to $kh_0 \ll 1$ – waves will be of small amplitude if the topography is of small amplitude, as measured by kh_0.

In the moving frame, the boundary condition becomes

$$\tilde{w} = Uh_0 k\cos(k(\xi + Ut)) \tag{4.46}$$

In other words, the topography appears to creates a time-dependent forcing on the lower boundary. Finally, note that this boundary condition should be applied at $z = h(x)$. However, if the topography is small, we can also apply the boundary condition at $z = 0$ instead by invoking a Taylor expansion ($\tilde{w}(z = h) = \tilde{w}(z = 0) + O(kh)$).

To summarize, we are now trying to solve

$$\frac{\partial^2}{\partial t^2}((\partial_{\xi\xi} + \partial_{zz})\phi) = -N^2\frac{\partial^2 \phi}{\partial \xi^2} \tag{4.47}$$

where ϕ is the streamfunction satisfying $\tilde{u} = \partial\phi/\partial z$ and $\tilde{w} = -\partial\phi/\partial\xi$, and where we apply the bottom boundary condition

$$\left.\frac{\partial\phi}{\partial\xi}\right|_{z=0} = -Uh_0 k\cos(k(\xi + Ut)) \tag{4.48}$$

The solutions, and the radiation condition

Let's seek solutions of the form

$$\phi = \Phi(z)e^{i(k_\xi\xi - \omega t)} \tag{4.49}$$

(we have to keep the dependence in z separate since the system is not infinite in that direction). We then have

$$-\omega^2(-k_\xi^2\Phi + \frac{d^2\Phi}{dz^2}) = N^2 k_\xi^2\Phi \tag{4.50}$$

which can be written as

$$\frac{d^2\Phi}{dz^2} = \left(1 - \frac{N^2}{\omega^2}\right)k_\xi^2\Phi \tag{4.51}$$

We see that there will be two types of solutions depending on the value of N^2/ω^2: oscillatory and exponential (evanescent). This is not surprising: as we saw earlier from the dispersion relation for internal waves in an infinite domain, gravity waves can only exist with frequencies that are smaller than N. The same applies here.

As with many problems of this kind, the eigenvalues (here, k_ξ and ω) are found by applying boundary conditions to the system. Since we defined ϕ as a complex variable, we have to rewrite the boundary condition (4.52) as

$$\Re\left\{\left.\frac{\partial\phi}{\partial\xi}\right|_{z=0}\right\} = -Uh_0 k\cos(k(\xi + Ut)) \tag{4.52}$$

which implies

$$\Re\left\{ ik_\xi\Phi(z)e^{ik_\xi\xi - i\omega t}\big|_{z=0}\right\} = -Uh_0 k\cos(k(\xi + Ut)) \rightarrow$$
$$-k_\xi\Phi_I(0)\cos(k_\xi\xi - \omega t) - k_\xi\Phi_R(0)\sin(k_\xi\xi - \omega t) = -Uh_0 k\cos(k(\xi + Ut))$$

where $\Phi_R(0) = \Re(\Phi(0))$ for short, and similarly for the imaginary part. This needs to be true for all time and all ξ. This can only happen if $k_\xi = k$, $\omega = -kU$, $\Phi_R(0) = 0$ and $\Phi_I(0) = Uh_0$. We can now plug this back into the governing equation, requiring that

$$\frac{d^2\Phi}{dz^2} = \left(k^2 - \frac{N^2}{U^2}\right)\Phi \tag{4.53}$$

with $\Phi(0) = iUh_0$ as a lower boundary condition.

Whether the wave field has an exponential or an oscillatory structure will therefore depend quite sensitively on the quantity

$$\mathrm{Fr} = \frac{k|U|}{N} \tag{4.54}$$

This is called a *Froude number*. If $\mathrm{Fr} \ll 1$, the background flow is called *subcritical* for the topography considered. If $\mathrm{Fr} \gg 1$, the flow is called *supercritical* for that topography. Note that Fr only depends on the strength of the stratification N and the spatial scale k^{-1} of the topography, but not on its height. Let's now look at both cases in turn.

If $\mathrm{Fr} \gg 1$, then $\Phi(z)$ has an exponential form. Keeping only the term that does not diverge as $z \to \infty$, we simply have

$$\Phi(z) = iUh_0 \exp(-mz) \text{ where } m = \sqrt{k^2 - \frac{N^2}{U^2}} \tag{4.55}$$

The horizontal and vertical velocities associated with this streamfunction are

$$\begin{aligned} \tilde{w} &= -\Re\left\{\frac{\partial \phi}{\partial \xi}\right\} = Uh_0 k \exp(-mz)\cos(k\xi - \omega t) = Uh_0 k \exp(-mz)\cos(kx) \\ \tilde{u} &= \Re\left\{\frac{\partial \phi}{\partial z}\right\} = Uh_0 m \exp(-mz)\sin(k\xi - \omega t) = Uh_0 m \exp(-mz)\sin(kx) \end{aligned} \tag{4.56}$$

We see that these quantities are actually stationary when written in the original frame of reference. In other words, the patterns created by the interaction of a steady flow with topography are, at linear order, also steady. We see that the horizontal flow is in phase with the topography, but the vertical flow is out of phase with it. Both decay exponentially with height on a vertical lengthscale $1/m$, which, for large-enough flow velocity U, is close to the typical horizontal lengthscale of the topography. In other words, the wave-induced flow is very much localized near the topography. This is illustrated in Figure 4.7 which shows the vertical velocity field above the ground, for a case where $m = k/2$.

If $\mathrm{Fr} \ll 1$, by contrast, the solutions are oscillatory:

$$\Phi(z) = A_+ e^{imz} + A_- e^{-imz} \text{ where } m = \sqrt{\frac{N^2}{U^2} - k^2} \tag{4.57}$$

This time, while we know that $A_+ + A_- = iUh_0$, we need another condition to find A_+ and A_- independently. This condition is called the *radiation condition*, and is basically a causality condition that sets the direction of propagation of energy. In this particular system, we expect the wave energy to come from the topography and propagate upwards[2]. To apply the radiation condition we must remember that energy propagates in the direction of the group velocity. Far from the topography, the latter is given by equation (4.22) where, in the notations of this section k_z is $\pm m$ (depending on the direction of propagation)

[2] If, somehow, there was a wavemaker at infinity radiating waves downward, then we would require that the energy propagates downward.

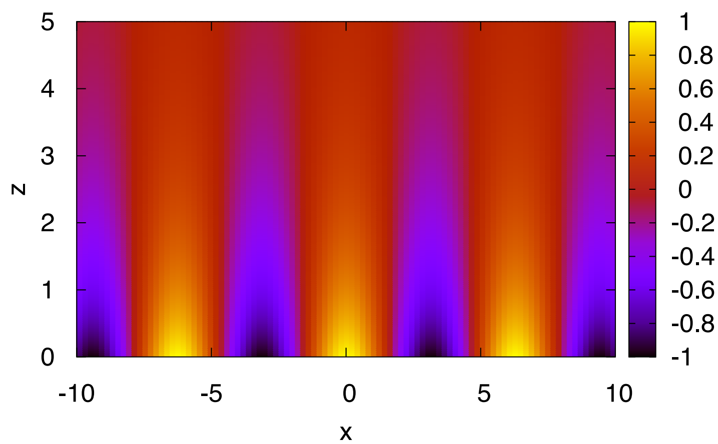

Figure 4.7: Solution for $\tilde{w}$ in the case Fr $\gg 1$, with $k = 1$ and $m = 1/2$.

and $k_x = k$. We see that for $\boldsymbol{c}_g \cdot \boldsymbol{e}_z > 0$, we have to have a negative value of m. This finally implies that

$$\Phi(z) = iUh_0 e^{-imz} \tag{4.58}$$

As before, we can calculate the actual velocities induced as

$$\begin{aligned} \tilde{w} &= -\Re\left\{\frac{\partial\phi}{\partial\xi}\right\} = Uh_0k\cos(k\xi - mz - \omega t) = Uh_0k\cos(kx - mz) \\ \tilde{u} &= \Re\left\{\frac{\partial\phi}{\partial z}\right\} = Uh_0m\cos(k\xi - mz - \omega t) = Uh_0m\cos(kx - mz) \end{aligned} \tag{4.59}$$

Again, the velocity field is stationary in the original reference frame. The vertical component of this velocity field is illustrated in Figure 4.8, in a case with $m = k/2$. Note that this time, $\tilde{u}$ and $\tilde{w}$ are in phase, which has important consequences in the next section.

These results explain a number of things. First, however, recall that we can now add solutions like this together to create the solution for any possible topography. As a result, it can easily be shown that if the topography is localized, then the perturbations will also be localized, around that topography. This then explains the formation of lenticular clouds very close to tall mountains that stand in the way of the wind. It also explains why these lenticular clouds (and other gravity-wave induced clouds) are always relatively stationary in comparison with the mountain range that generates them. Finally this also explains why in some cases the cloud pattern has a single cell in the vertical direction, while in others, multiple cells exist stacked on top of each other – this merely depends on the value of k, N and U. Similar features exist in the bottom of the ocean, but are not as obviously spectacular (since we usually can't see them).

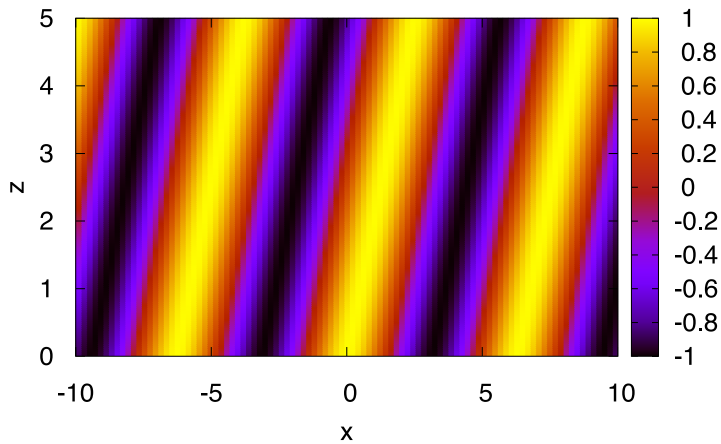

Figure 4.8: Solution for $\tilde{w}$ in the case Fr $\ll 1$, with $k = 1$ and $m = 1/2$. Note how the lines of constant $\tilde{w}$ are tilted in the direction of the flow U.

Effect of the waves on the background flow

We expect the bottom topography to create a drag on the flow passing above it, and to gradually slow it down. Let's now study this effect. This is an example of *wave-mean flow interaction*. We shall revisit this issue in the next section as well.

To study the effect of the waves on the mean flow, let's consider again the momentum equation, but this time keep all the nonlinear terms. Ignoring viscosity again, we have in the horizontal direction,

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} + U)\frac{\partial \tilde{u}}{\partial x} + \tilde{w}\frac{\partial \tilde{u}}{\partial z} = -\frac{\partial \tilde{p}}{\partial x} \tag{4.60}$$

We now use the mass continuity equation ($\nabla \cdot \tilde{\boldsymbol{u}} = 0$) to rewrite this as

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\partial}{\partial x}(\tilde{u}^2) + \frac{\partial}{\partial z}(\tilde{w}\tilde{u}) = -\frac{\partial \tilde{p}}{\partial x} \tag{4.61}$$

Next, we average this equation horizontally, assuming periodicity in x, and multiply by ρ_m to get

$$\rho_m \frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial z}\overline{\rho_m \tilde{w}\tilde{u}} = 0 \tag{4.62}$$

This shows that the rate of change in the average horizontal flow is *quadratic* in the wave amplitude. For small amplitude waves, this term is small (and was indeed neglected at linear order). But if we don't neglect it, then this term tells us how the mean flow is slowed down by the waves caused by the passage over the topography. Let's evaluate the *momentum flux* $\rho_m\overline{\tilde{w}\tilde{u}}$.

In the case where Fr $\gg 1$, then

$$\rho_m\overline{\tilde{w}\tilde{u}} = \rho_m U^2 h_0^2 km \exp(-2mz)\overline{\cos(kx)\sin(kx)} = 0 \tag{4.63}$$

This implies that the perturbation flow induced by the interaction between the background and the topography has no momentum flux at all, and will therefore not have any effect on the mean flow whatsoever! This somewhat counter-intuitive result is nevertheless correct, and has been verified experimentally.

In the case where Fr $\ll 1$, then

$$\rho_m \overline{\tilde{w}\tilde{u}} = \rho_m U^2 h_0^2 km \overline{\cos^2(kx - mz)} = \frac{\rho_m}{2} U h_0^2 k \sqrt{U^2 k^2 - N^2} \tag{4.64}$$

In this limit the waves transport net momentum (as well as energy) away from the bottom topography. The rate at which this happens depends quadratically on the amplitude of the topography, h_0 – this is not surprising, as we expect that the larger h_0 is, the stronger waves are driven. It also increases with U, which again is not surprising. Everything else being equal, we also find that the momentum flux is larger if the topography has a smaller wavelength (larger k), or if the fluid is less stratified (smaller N). The reason for the former is also fairly intuitive : the more "bumps" there are, the larger the drag on the background flow. The latter is more subtle, but is related to the vertical displacement – the larger N, the smaller the vertical displacement caused by the wave (larger m), hence the smaller the momentum flux.

However, note that $\rho_m \overline{\tilde{w}\tilde{u}}$ is independent of z in this idealized model, which implies that the horizontal mean flow *does not evolve with time* (see equation (4.62)). In other words, the waves transport momentum upward without affecting the mean flow. Of course, in a real system the momentum flux is unlikely to be constant: the background density, buoyancy frequency, and mean flow velocity usually vary slowly with height and time. Also, waves dissipate (either by thermal diffusion or viscous dissipation) or can steepen nonlinearly if they have large enough amplitudes. All of these effects can cause the momentum flux to depend on z, and therefore change the mean flow. This is the wave-induced drag mentioned earlier. It is interesting to note that the drag is essentially non-local, i.e. the wave can propagate a long way before dissipating and exchanging momentum with the background, so the location where the mean flow slows down could be very far away from the topography! This actually occurs in the Earth's atmosphere, and has been proposed as a mechanism to slow down the cores of stars (via gravity waves that are generated at the bottom of the convection zone).