

Chapter 1

The governing equations of fluid dynamics

In this series of lectures we will study waves and instabilities in fluid flows, with particular applications to geophysical and astrophysical systems, and paying particular attention to the various mathematical methods that can be used to characterize their properties. The course will be divided into two distinct parts, the first 5 weeks being dedicated to the study of waves, and the next 5 weeks to the study of instabilities. In the time available, it will not be possible to be comprehensive. However, we will study a few representative examples of waves and instabilities, and learn various tools to analyze them. The goal of the course is to give you enough background in the field to make you comfortable approaching any new problem you may encounter in the course of your future research.

In this first Chapter, we will briefly discuss the governing equations of fluid dynamics, their derivation and their meaning. For an excellent introduction to this topic, see the book by Rick Salmon, “Lectures on Geophysical Fluid Dynamics”, Chapter 1.

The equations of fluid dynamics were first derived by Navier and Stokes in the early-to-mid 1800s. This taking place long before atoms and molecules were discovered, fluids were viewed as a continuum, and modeled as a collection of adjacent macroscopic “fluid elements” that are governed by Newton’s laws. These elements are subject to various forces (gravity, viscous friction, etc.) that control their motion, and must satisfy appropriate conservation laws (mass, momentum, energy, etc..).

In the late 1800s/early 1900s, however, the discovery of the atom revealed the true nature of a fluid to be a vast collection of interacting atoms, ions and electrons, or molecules. Each of these particles is subject to its own equation of motion, governed by body forces such as gravity, but also by local electrostatic forces induced by nearby particles, collisions, etc. The equations of fluid

dynamics were then re-derived by considering statistical averages of the equations that govern the motion of each individual particle within the fluid. The Navier-Stokes equations can be recovered in some limits, but this approach is more general and can be used to model more “exotic” fluids such as relativistic and/or strongly magnetized plasmas.

These two approaches, macroscopic and microscopic, are still in use today, each preferred over the other depending on the application of interest. The latter yields a set of equations that are mathematically exact, but involve progressively higher statistical moments of the equations that can only be modeled using additional theories or assumptions concerning the molecular interactions. The former is not exact (and requires assumptions about the mathematical form of some of the material forces within the fluid), but its interpretation strongly benefits from its origin in standard Newtonian mechanics. In what follows, we will only consider the traditional “macroscopic” approach to the derivation of the fluid equations.

1.1 Mass conservation, Eulerian and Lagrangian descriptions of a fluid

1.1.1 Eulerian description

In what follows, we consider the fluid as a continuum. At each point in space (x, y, z) or time (t) , it can be characterized by its velocity field, the 3D vector $\mathbf{u}(x, y, z, t)$, and its thermodynamical properties: its density $\rho(x, y, z, t)$, its pressure $p(x, y, z, t)$, etc... This is called the *Eulerian* description of the fluid. In many cases, the Eulerian description of the fluid yields a simple, easy-to-understand view of the motion of the fluid.

For instance, consider the following 2D vector field:

$$\mathbf{u}(x, y, t) = U_0 \exp\left(-\frac{y^2}{2}\right) \mathbf{e}_x \quad (1.1)$$

This corresponds to a jet that flows in the x -direction, and has a velocity that varies in y but is independent of x and t . The flow field

$$\mathbf{u}(x, y, t) = \frac{U_0}{\sqrt{t}} \exp\left(-\frac{(x-2t)^2}{2t}\right) \mathbf{e}_y \quad (1.2)$$

corresponds to another jet, in the y direction this time, whose width and position both increase with time, but whose amplitude decreases.

Similarly, the density field

$$\rho(x, y, t) = \rho_0 \sin(x) \cos(y) \quad (1.3)$$

corresponds to a spatially periodic modulation that is independent of time, while

$$\rho(x, y, t) = \rho_0 \exp(-t/\tau) \sin(x) \cos(y) \quad (1.4)$$

corresponds to the same modulation, but this time the amplitude decays with time.

In other words, the Eulerian description is useful and intuitive for anyone with experience in visualizing 2D or 3D functions or vector fields. However, it is not necessarily the simplest one in terms of the derivation and interpretation of the equations of fluid dynamics. To see this, let us first study mass conservation.

1.1.2 Mass conservation

The notion of mass conservation is quite simple. It states that, in a given region of space D (that is fixed and independent of time), the amount of mass in that region can only change if mass is brought in, or taken out of it. This can be expressed mathematically as

$$\frac{\partial}{\partial t} \int_D \rho(\mathbf{x}, t) d^3 \mathbf{x} = - \int_{\partial D} \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t) \cdot \hat{\mathbf{n}}(\mathbf{x}) d^2 \mathbf{x} \quad (1.5)$$

where the first term is the change in the total mass inside the domain D and the second term is the integral over the surface of D of the component of the mass flux $(\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t))$ normal to the surface. Since the unit normal to the surface $\hat{\mathbf{n}}$ is defined outward, $-\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t) \cdot \hat{\mathbf{n}}(\mathbf{x})$ is positive if material is going into D and negative if material is going out of D .

Using the divergence theorem and using the fact that D is independent of time, we can rewrite this as

$$\int_D \frac{\partial}{\partial t} \rho(\mathbf{x}, t) d^3 \mathbf{x} = - \int_D \nabla \cdot (\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)) d^3 \mathbf{x} \quad (1.6)$$

Since this has to hold for *any* domain D , one can take the limit where the volume of D tends to 0, and obtain the *local* conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1.7)$$

which must be valid at every point in space or time. This is the *mass conservation equation*, and is, as written, expressed in Eulerian coordinates.

1.1.3 Properties of mass conservation: advection vs. compression/expansion

Let's consider a 1D version of the mass conservation equation. In this case, both ρ and u are function of x and t only. We then have

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = \frac{\partial \rho}{\partial t} + u(x, t) \frac{\partial \rho}{\partial x} + \rho(x, t) \frac{\partial u}{\partial x} = 0 \quad (1.8)$$

Separating the second term into two parts reveals two contributions to any change in ρ : one that corresponds to advection, and one that corresponds to the effect of compression or expansion. To see this, we now consider both in turn.

Let's first consider a "divergence-free" ($\nabla \cdot \mathbf{u} = 0$) velocity field \mathbf{u} , which in 1D simply means that $\partial u / \partial x = 0$. This corresponds to a homogeneous flow $u(x, t) = u_0(t)$. For simplicity, let's just choose a constant u_0 although the following argument can be generalized to arbitrary $u_0(t)$. The mass conservation equation then becomes

$$\frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = 0 \quad (1.9)$$

In this particular example, the solution is very simple and takes the form

$$\rho(x, t) = \rho_0(x - u_0 t) \quad (1.10)$$

where $\rho_0(x) = \rho(x, 0)$ is the initial density distribution. In other words, the density field is merely moved at constant velocity left or right (according to the sign of u_0), without change of form. This statement can easily be generalized if u_0 is a function of time, or even in higher dimensions.

An equivalent way of looking at this problem is to consider that, at time $t = 0$, the x -axis is discretized into a set of small elements of size dx , each of which contains a total mass $\rho(x, 0)dx$. Let's focus for instance a given fluid element initially located at $x = x_0$. If it is moving at the constant velocity u_0 , then at a later time t it can be found at the position $x_e(t) = x_0 + u_0 t$. Given (1.10), we can then ask what the density of the fluid *following a moving element* is. This is given by

$$\rho(x_e(t), t) = \rho_0(x_e(t) - u_0 t) = \rho_0(x_0) \quad (1.11)$$

In other words, we find that the parcel moves around without any change in its density! This is a general property of divergence-free flow fields, that can be generalized to more than 1 dimension, and to cases where \mathbf{u} is not constant.

Noting this, it is clear that there must be a simpler way of rewriting the equation for mass conservation that emphasizes this property. And indeed, there is – as long as one rewrites it in a coordinate system that follows the fluid element along its trajectory! This idea is called the *Lagrangian view* of a system, and is quite general. First note that creating a coordinate system is merely creating labels. In a Eulerian coordinate system, each element is labeled by time and by its current position. There is nothing to stop us, however, labeling a fluid element by time and by its *initial* position instead. To go from one coordinate system to the other, that is, from (x, t) to (x_0, τ) , we use the transformation $t = \tau$ and $x = x_e(t) = x_0 + u_0 \tau$. The advantage of the new coordinate system is that, as long as x_0 is held constant, we are always considering the *same* fluid element. Hence the partial derivative with respect to τ (holding x_0 constant) is exactly the derivative along a trajectory that we were interested in deriving. Using the coordinate transformations above we get

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial t}{\partial \tau} \frac{\partial \rho}{\partial t} + \frac{\partial x}{\partial \tau} \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = 0 \quad (1.12)$$

In other words, mass conservation written in a Lagrangian coordinate system, for a divergence-free flow field, simply becomes

$$\frac{\partial \rho}{\partial \tau} = 0 \quad (1.13)$$

which expresses the fact that the density ρ of a given fluid element is independent of time along its trajectory, as discussed earlier. For anyone with experience with the method of characteristics, it is clear that switching to a Lagrangian approach in this example is completely equivalent to it.

Let's now go back to the original mass conservation equation and consider a general velocity field $u(x, t)$ that is not necessarily divergence free. The solution $\rho(x, t)$ is no longer as simple, although can in principle be obtained using the method of characteristics (cf. AMS212A) if required. The interpretation of the equation is also rather obscure, when viewed in an Eulerian frame of reference, that is, at fixed x and t . However, it turns out that we can gain insight into the problem by looking at it, as we did earlier, from a reference frame that is moving with the fluid elements.

Let's move again into the coordinate system (x_0, τ) . This time, we can't necessarily write down a simple expression for x as a function of x_0 and t as we did before, since u could be a function of both x and t instead of being constant. However, we can still calculate the current position of an element that started at x_0 as

$$x_e(t) = x_0 + \int_0^t u(x_e(t'), t') dt' \quad (1.14)$$

and consider the new coordinate system (x_0, τ) such that $\tau = t$ and $x = x_e(\tau)$. The derivative of ρ along a trajectory then becomes

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial t}{\partial \tau} \frac{\partial \rho}{\partial t} + \frac{\partial x}{\partial \tau} \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial t} \frac{\partial t}{\partial \tau} + u(x, t) \frac{\partial \rho}{\partial x} = -\frac{\partial u}{\partial x} \rho \quad (1.15)$$

This implies that if u increases with x , the fluid element's density must decrease with time, while when u decreases with x , the fluid element's density increases. To understand physically why this is the case, simply note that if u is positive, and increases with x , fluid elements progressively move away from each other as time goes by – the fluid expands. Meanwhile, if u is positive, and decreases with x , fluid elements move closer to one another, and the fluid is compressed. Similar statements can be made if u is negative. We then see that the expansion or compression of a fluid element results in a change of density, as expected.

To summarize, we can see that while the Eulerian description is useful to visualize the flow, the expression of the mass conservation law is simpler and physically more intuitive when expressed in Lagrangian coordinates.

1.1.4 Lagrangian description of a fluid in 3D

The concept of Lagrangian coordinates in 3D can be easily be generalized from the 1D case. Using exactly the same notations as before, each fluid element is

labeled in the Lagrangian coordinate system according to their original position (x_0, y_0, z_0) at $t = 0$. The elements moves around following the 3D velocity field \mathbf{u} , and end up at the position $(x_e(t), y_e(t), z_e(t))$ at time t . The functions $x_e(t)$, $y_e(t)$ and $z_e(t)$ are, as in the 1D case, time-integrals of the velocity field, and conversedly, we have that $\mathbf{u} = (\partial x_e/\partial t, \partial y_e/\partial t, \partial z_e/\partial t)$. Hence the derivative of any function *following the fluid element* is given by

$$\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial t} + \frac{\partial x}{\partial \tau} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \tau} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \tau} \frac{\partial f}{\partial z} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f \quad (1.16)$$

The mass conservation, for instance, can be written as

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) - \rho \nabla \cdot \mathbf{u} = -\rho \nabla \cdot \mathbf{u} \quad (1.17)$$

In other words, the density of a fluid element can only change as it moves around if it enters a region where the flow field has a non-zero divergence. A positive divergence means that the flow expands, and the density decreases, while a negative divergence implies compression, and the density increases.

In all that follows, we will use the more conventional notation for the derivative of a quantity following the fluid to be $Df/Dt = \partial f/\partial t + \mathbf{u} \cdot \nabla f$. Mass conservation then becomes

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \quad (1.18)$$

1.2 Momentum conservation

Newton's second law states that the mass times the acceleration of any massive object is equal to the sum of all forces applied to it. Applying this to a fluid element we first need to define its acceleration. This is given by the time-derivative of its velocity *following the fluid element*, so Newton's law becomes

$$\rho \frac{D\mathbf{u}}{Dt} = \sum_i \mathbf{F}_i \quad (1.19)$$

The various forces acting on the fluid depend on the situation considered but always include, at the very least, the effect of pressure gradients and viscous friction, expressed as

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \mathbf{\Pi} \quad (1.20)$$

where p is the pressure, and $\mathbf{\Pi}$ is the viscous stress tensor. In Cartesian coordinates, the latter is given by

$$\Pi_{ij} = \left(\kappa - \frac{2}{3}\mu \right) \delta_{ij} \nabla \cdot \mathbf{u} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1.21)$$

where μ and κ are two scalars that depend on the microscopic properties of the fluid. It can also be written, in more general vector form, as

$$\mathbf{\Pi} = \left(\kappa - \frac{2}{3}\mu \right) (\nabla \cdot \mathbf{u}) \mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \quad (1.22)$$

1.3. THE EQUATION OF STATE, AND THERMAL ENERGY CONSERVATION⁷

where \mathbf{I} is the identity matrix. Other common possible forces include buoyancy (gravity) and Lorentz (magnetic) forces for instance.

1.3 The equation of state, and thermal energy conservation

The following sections are adapted from the book “Fluid Mechanics” by Kundu and Cohen.

So far, we have two equations (the momentum equation and the mass conservation equation) for three unknowns (counting the momentum equation as 1 equation, and the whole vector field as 1 unknown): p , ρ and \mathbf{u} . We are clearly still missing some information to fully characterize the behavior of the fluid. To relate pressure and density, one needs to appeal to the field of thermodynamics.

Thermodynamics was developed in the 1800s to study the macroscopic properties of liquids or gases. A fluid¹ in a container of volume V , at temperature T , has a pressure p and a density ρ . These quantities are not independent, however. When mass is conserved, $\rho = M/V$ where M is the total mass of material inside of V . But we also know from everyday observations that pressure increases when a gas is heated at constant volume (that’s what happens in a pressure cooker), or when the volume decreases at constant temperature (that’s what happens in a piston). So p , ρ and T must be related to one another – the question is *How?*

The relationship between thermodynamical variables is called “the equation of state”. There are many equations of state for a given fluid, but they are all equivalent: they can be written as $p = p(\rho, T)$ or $\rho = \rho(T, p)$, but they can also involve other thermodynamic properties of the fluid, such as its internal energy, its entropy, etc... An important property of thermodynamics, however, is that the *complete* state of a fluid can be fully described by knowing *only two* of these variables (for a single-component fluid), hence an equation of state is always of the form $X = f(Y, Z)$. The equation of state can be viewed as the third equation of fluid dynamics next to momentum and mass conservation.

For gases, a well-known equation of state is that of a perfect gas:

$$p = R\rho T \tag{1.23}$$

where R is a constant that is specific to each gas (called “the gas constant”). The value of R depends on the molecular mass of the gas m according to

$$R = \frac{R_u}{m} \tag{1.24}$$

¹In all this course, we will usually use the terminology *fluid* to describe either a *liquid* or a *gas* – both being adequately described by the equations of fluid dynamics. However, the thermodynamics of liquids and gases are quite different, so in this section they will be referred to specifically by their nature when relevant.

where R_u is the *universal gas constant*, and m in this equation is the molecular mass of the gas (i.e. the total mass of exactly 1 mole of gas). This equation is derived from the kinetic theory of gases assuming that electrostatic forces between the particles are negligible. Other equations of state exist when these forces become important (notably the Van der Waals equation of state). In this course, we will always assume that a gas can be described by a perfect gas equation of state, however.

The equation of state of liquids, by contrast, is usually very complicated (even though it only depends on two variables), and difficult to construct from first principles because the properties of liquids depend sensitively on intramolecular forces. Generally, however, the equation of state takes the approximate form

$$\rho = \rho(T) \tag{1.25}$$

which is independent of pressure – liquids are, to a great degree of approximation, incompressible so the pressure does not play any role in the equation of state.

In both cases, for gases and liquids, we see that the equation of state depends explicitly on the temperature T . This means that, except in very special circumstances, we will need to introduce a fourth equation for the evolution of temperature. This final equation can be derived from the conservation of total energy, which also depends on the thermodynamics of the fluid. In the interest of time, we will not be deriving it here from “first principles”, but merely state it and discuss its consequences. For more detail, see Kundu & Cohen.

The thermal energy conservation equation can in all generality be written as

$$\rho \frac{De}{Dt} = -p \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q} + Q + \phi \tag{1.26}$$

where e is the *internal energy* of the gas/fluid, Q is the net local heating/cooling term (that could come from local nuclear reactions in a star, or from radiative cooling in an optically thin atmosphere), \mathbf{q} is the local heat flux and ϕ is the energy dissipation rate due to viscosity. This equation can be derived from the First Law of Thermodynamics and from the mechanical energy conservation equation (which comes from dotting the momentum equation with \mathbf{u}). To interpret it, note that e measures the energy stored in the microscopic motion of the particles, that is, their rms kinetic energy plus any energy coming from the vibrations or rotation of molecules (if we are thinking of a liquid or a molecular gas). Any change in e can only be due to effects that change these microscopic motions: this includes heating or cooling (either direct heating from Q and ϕ , or from the convergence of the heat flux), or by compression (which forces the components of the gas closer to one another and therefore affects their collisions, kinetic energy or interactions).

For a perfect gas, there is a very simple relation between e and T :

$$e = c_v T \tag{1.27}$$

where c_v is a constant that only depends on the nature of the constituents of the gas. In that case, the thermal energy equation becomes

$$\rho c_v \frac{DT}{Dt} = -p \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q} + Q + \phi \quad (1.28)$$

Also, note that except in particular circumstances we have that

- The viscous dissipation is negligible (except when shocks are present), so ϕ can be ignored
- The heat flux is usually proportional to the local temperature gradient via Hook's Law:

$$\mathbf{q} = -k \nabla T \quad (1.29)$$

where k is called the “thermal conductivity”, unless the medium is very optically thin, or a strongly magnetized plasma.

In these cases, the thermal energy simply becomes

$$\rho c_v \frac{DT}{Dt} = -p \nabla \cdot \mathbf{u} + \nabla \cdot (k \nabla T) \quad (1.30)$$

This equation turns out to hold also for liquids under the same two assumptions, because they satisfy $De/Dt = c_v DT/Dt$ although in that case c_v is not necessarily constant. In addition, the term $-p \nabla \cdot \mathbf{u}$ drops out since liquids are incompressible. In that case we have

$$\rho c_v \frac{DT}{Dt} = \nabla \cdot (k \nabla T) \quad (1.31)$$

1.4 Entropy and adiabaticity

Another thermodynamic variable is the specific entropy s , defined through the Second Law of Thermodynamics. It is related to the internal energy and other variables via

$$\frac{De}{Dt} = T \frac{Ds}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} \quad (1.32)$$

Plugging (1.18) and (1.26) into this equation, we get

$$\rho T \frac{Ds}{Dt} = -\nabla \cdot \mathbf{q} + Q + \phi \quad (1.33)$$

This implies that entropy changes whenever heat is added or removed from the system – via any of the following processes: convergence or divergence of heat flux, local heat source/sink and viscous dissipation. This equation is often used as an alternative to the thermal energy equation.

Note how entropy is conserved in any system where there is no heat exchange. Such systems are called *adiabatic*, and satisfy:

$$\frac{Ds}{Dt} = 0 \quad (1.34)$$

For a perfect gas, it can be shown via kinetic theory or thermodynamics that adiabaticity implies

$$p = f(s)\rho^\gamma \quad (1.35)$$

where $f(s)$ is some function of the specific entropy (which usually doesn't need to be specified), and γ is called the *adiabatic index*. Hence an alternative way of stating the conservation of entropy is

$$\frac{Dp}{Dt} = \gamma \frac{p}{\rho} \frac{D\rho}{Dt} \quad (1.36)$$

This also implies that

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \frac{p}{\rho} = \gamma RT \quad (1.37)$$

This quantity will shortly be revealed to be the square of the speed of sound if sound waves are assumed to propagate adiabatically.

For liquids, adiabaticity simply implies that

$$\frac{DT}{Dt} = 0 \quad (1.38)$$

which can also be written as

$$\frac{D\rho}{Dt} = 0 \quad (1.39)$$

using the equation of state. Note that this final equation is consistent with mass conservation, and the fact that $\nabla \cdot \mathbf{u} = 0$, so for adiabatic liquids the thermal energy equation is redundant.