Homework 3

January 25, 2016

Graduate students: Either do Problem set 1 on the Dedalus page of the website, or do the following two coursework problems.

Undergraduate students doing an independent study: Homework 3 is optional, counts for extra credit.

1 Problem 1

Try to understand the derivation in the lecture notes to show, and show *step by step*, that the amplitude equation (2.8) of the Chapter 2 part 3 notes is indeed equivlent to the energy conservation equation (2.47). Note that the lectures give you the main steps of the calculation, but you have to fill in the gaps.

2 Problem 2

Consider the governing equation for internal gravity waves (equation 4.7), in the case where N is a slowly varying function of $\boldsymbol{X}$ and the slow time τ. Assume a wave packet solution of the form

$$\phi = A(\boldsymbol{X}, \tau) e^{i\theta(\boldsymbol{x}, t)} \tag{1}$$

with $\boldsymbol{k} = \nabla\theta$ and $\omega = -\partial\theta/\partial t$.

Starting from these assumptions only, prove equations (4.29), (4.30), (4.31) and (4.37) of the lecture notes. The first 3 are not too difficult, but the last one is quite hard. You may want to first show that the amplitude equation satisfies:

$$2\omega \frac{\partial k^2}{\partial \tau} A + k^2 \frac{\partial \omega}{\partial \tau} A + 2\omega k^2 \frac{\partial A}{\partial \tau} - \omega^2 \left(2\boldsymbol{k} \cdot \nabla A + A \nabla \cdot \boldsymbol{k}\right) = -\left[N^2 \frac{\partial k_x}{\partial X} A + k_x \frac{\partial N^2}{\partial X} A + 2k_x N^2 \frac{\partial A}{\partial X} \right] \tag{2}$$

Then, using the standard trick of multiplying by the complex conjugate of A, and rearranging terms, write this equation as

$$\frac{\partial}{\partial \tau}(|A|^2 k^2) + \nabla \cdot \left(\boldsymbol{c}_g k^2 |A|^2\right) = \text{stuff} \tag{3}$$

The whole problem then reduces to showing that “stuff” is the RHS of (4.37). Equation (4.22) will turn out to be very useful.