I. 2 Roots of polynomials

Many problems in applied mathematics lead to the derivation of a polynomial equation whose roots we are interested in

- The characteristic equation of a linear ODE with constant coefficients
- Finding steady states of a dynamical system whose RHS is polynomial (cf bifurcation theory)
- Stability analyses in many fluid systems

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Here we will assume that the polynomial has already been non-dimensionalized, and consider the simple example where it is of the form

$$\epsilon x^2 + x - 1 = 0 \qquad \epsilon \text{ is small.}$$

a. The iterative method

As in the previous example, let's assume we forgot how to solve a quadratic, & try to solve this one iteratively.

The first attempt, setting $\epsilon = 0$, yields

$$x^{(0)} - 1 = 0 \Rightarrow x^{(0)} = 1$$

At the next order, we use the previous solution wherever the term is multiplied by ϵ: this yields

$$\epsilon [x^{(0)}]^2 - 1 + x^{(1)} = 0 \Rightarrow$$

$$x^{(1)} = 1 - \epsilon (1)^2 = 1 - \epsilon$$

Similarly, we do

$$\epsilon\left[x^{(n-1)}\right]^2 + x^{(n)} - 1 = 0$$

from here on, so to the next orders

- $\epsilon\left[x^{(1)}\right]^2 + x^{(2)} - 1 = 0$

$$\Rightarrow x^{(2)} = 1 - \epsilon\left[1-\epsilon\right]^2 = 1 - \epsilon + 2\epsilon^2 - \epsilon^3$$

- $\epsilon\left[x^{(2)}\right]^2 + x^{(3)} - 1 = 0$

$$\Rightarrow x^{(3)} = 1 - \epsilon\left[1-\epsilon+2\epsilon^2-\epsilon^3\right]^2$$
$$= 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + 6\epsilon^4 - 6\epsilon^5 + 4\epsilon^6 - \epsilon^7$$

and so forth.

This is a very simple method, but this time it's no longer clear that it is completely self-consistent.

Indeed we have

$$x^{(1)} = 1 - \epsilon$$
$$= x^{(0)} - \epsilon$$

→ looks OK, the correction term $-\epsilon$ is small compared with $x^{(0)}$

$$x^{(2)} = 1 - \epsilon + 2\epsilon^2 - \epsilon^3$$
$$= x^{(1)} + 2\epsilon^2 - \epsilon^3$$

→ again looks OK, the correction terms is small compared to $x^{(1)}$

$$x^{(3)} = 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + 6\epsilon^4 - 6\epsilon^5 - 4\epsilon^6 - \epsilon^7$$
$$= x^{(2)} - 4\epsilon^3 + \dots - \epsilon^7$$

but this also contains an ϵ^3 term

So this means that the terms neglected in the $x^{(2)}$ solution are of the same order as the ones kept ... how do we know this won't happen again for $x^{(4)}, x^{(5)}, \dots$

6. Asymptotic sequence assumption

In this particular case, it's better to use an asymptotic sequence expansion from the start. As described in I.1, they are better mathematically-justified than the iterative method. The iterative method on the other hand is useful because it suggests what expansion to use.

Here the sequence is clearly in orders of ϵ $\Rightarrow$

let $\quad x = s_0 + \epsilon s_1 + \epsilon^2 s_2 + \epsilon^3 s_3 + \cdots$

then

$$\epsilon\left[s_0 + \epsilon s_1 + \epsilon^2 s_2 + \epsilon^3 s_3 \cdots\right]^2 + \left[s_0 + \epsilon s_1 + \epsilon^2 s_2 + \epsilon^3 s_3 + \cdots\right] - 1 = 0$$

$\Rightarrow$ order by order we get:

to $O(\epsilon^0)$: $\quad s_0 - 1 = 0 \quad \Rightarrow \quad s_0 = 1$

to $O(\epsilon^1)$: $\quad s_0^2 + s_1 = 0 \quad \Rightarrow \quad s_1 = -s_0^2 = -1$

to $O(\epsilon^2)$: $\quad 2s_0 s_1 + s_2 = 0 \quad \Rightarrow \quad s_2 = -2s_0 s_1 = 2$

to $O(\epsilon^3)$: $\quad 2s_0 s_2 + s_1^2 + s_3 = 0 \quad \Rightarrow \quad s_3 = -2s_0 s_2 - s_1^2 = -4 - 1 = -5$

$\vdots$

$\rightarrow$ we now see that, formally,

$$x = 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \underbrace{\cdots}_{\text{no terms here contain } \epsilon^3}$$

Again this is a great, efficient way of finding the asymptotic sequence provided we know what the correct powers of ϵ to use are.

However, we already see in this example that there is a problem. This gives one of the solutions of the quadratic. What happened to the other?

c. Exact solutions

Since we can, actually, solve this, let's do it & try to understand why we didn't find both solutions.

$$\epsilon x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+4\epsilon}}{2\epsilon}$$

Expanding the + solution in powers of ϵ we get

$$x \simeq \frac{1}{2\epsilon}\left(\cancel{-1} + \left(\cancel{1} + \frac{4}{2}\epsilon - \frac{1}{4}\cdot\frac{(4\epsilon)^2}{2} + \frac{3}{8}\frac{(4\epsilon)^3}{6} - \frac{15}{16}\frac{(4\epsilon)^4}{24}\cdots\right)\right)$$

$$= 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 \ldots \quad \rightarrow \text{this is the solution we already got!}$$

Expanding the − solution in powers of ϵ:

$$x \simeq \frac{1}{2\epsilon}\left(-1 - \left(1 + \frac{4}{2}\epsilon - \frac{1}{4}\frac{(4\epsilon)^2}{2} + \cdots\right)\right)$$

$$\simeq -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 \ldots$$

↑ this is a new term we had not accounted for in our ansatz for the asymptotic expansion!

The problem comes from the fact that we had assumed the solution is a regular expansion in the small parameter ϵ, that is, that we can get it by assuming that it is a small perturbation of the solution of the equation in which $\epsilon = 0$. But that can't be the case:

If $\epsilon \neq 0$: $\epsilon x^2 + x - 1 = 0$ has $\underline{\underline{2}}$ solutions

if $\epsilon = 0$: $x - 1 = 0$ has $\underline{\underline{1}}$ solution

Clearly, the limit $\epsilon \to 0$ is singular in the sense that it completely changes the nature of the equation and therefore of its solution(s).

16.

d Detour: What if ϵ had been on a different term in the polynomial.

Consider instead $x^2 + \epsilon x - 1 = 0$

(a similar equation, but ϵ now multiplies the term in x instead of the highest-order term).

Let's first try an 'iterative method':

Setting $\epsilon = 0$ we get $[x^{(0)}]^2 - 1 = 0 \Rightarrow x^{(0)} = \pm 1$

$\rightarrow$ two solutions!

Let's "follow" the $\oplus$ solution first. As before,

$$[x^{(1)}]^2 + \epsilon x^{(0)} - 1 = 0$$

$$\rightarrow [x^{(1)}]^2 = 1 - \epsilon x^{(0)} = 1 - \epsilon$$

$$\rightarrow \quad x^{(1)} = \pm\sqrt{1-\epsilon} \quad \rightarrow \text{another 2 solutions!}$$

$$\simeq \pm\left(1 - \frac{\epsilon}{2} + \cdots\right)$$

Since we had taken the $\oplus$ solution at the start we need to continue with that, so we can write $x^{(1)} = x^{(0)} - \frac{\epsilon}{2} + \cdots$

At the next step:

$$[x^{(2)}]^2 + \epsilon x^{(1)} - 1 = 0$$

$$[x^{(2)}]^2 = 1 - \epsilon x^{(1)} = 1 - \epsilon\sqrt{1-\epsilon}$$

$$x^{(2)} = \pm\sqrt{1 - \epsilon\sqrt{1-\epsilon}}$$

$$\simeq \pm\left[1 - \frac{\epsilon}{2}\sqrt{1-\epsilon} + \cdots\right]$$

$$\simeq \pm\left[1 - \frac{\epsilon}{2}\left(1 - \frac{\epsilon}{2} + \cdots\right) + \cdots\right]$$

pick $\oplus$ solution for consistency

$$\simeq \oplus\left(1 - \frac{\epsilon}{2} + \text{term in } \epsilon^2 \cdots\right)$$

And similarly for the negative root.

→ we see that a good ansatz for an asymptotic sequence here is

$$x = S_0^{(0)} + \epsilon S_1 + \epsilon^2 S_2 + \dots$$

let's try that:

$$(S_0 + \epsilon S_1 + \epsilon^2 S_2 + \dots)^2 + \epsilon(S_0 + \epsilon S_1 + \epsilon^2 S_2 + \dots) - 1 = 0$$

$$\Rightarrow \begin{cases} \text{to } O(\epsilon^0): & S_0^2 - 1 = 0 \Rightarrow S_0 = \pm 1 \\ \text{to } O(\epsilon^1): & 2S_0 S_1 + S_0 = 0 \Rightarrow S_1 = -\frac{1}{2} \\ \text{to } O(\epsilon^2): & 2S_0 S_2 + S_1^2 + S_1 = 0 \\ & \Rightarrow S_2 = \dfrac{-S_1 - S_1^2}{2S_0} = \pm \dfrac{\frac{1}{2} - \frac{1}{4}}{2} \\ & = \pm \frac{1}{8} \end{cases}$$

So we have 2 solutions:

$$\begin{cases} x = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \dots \\ x = -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \dots \end{cases}$$

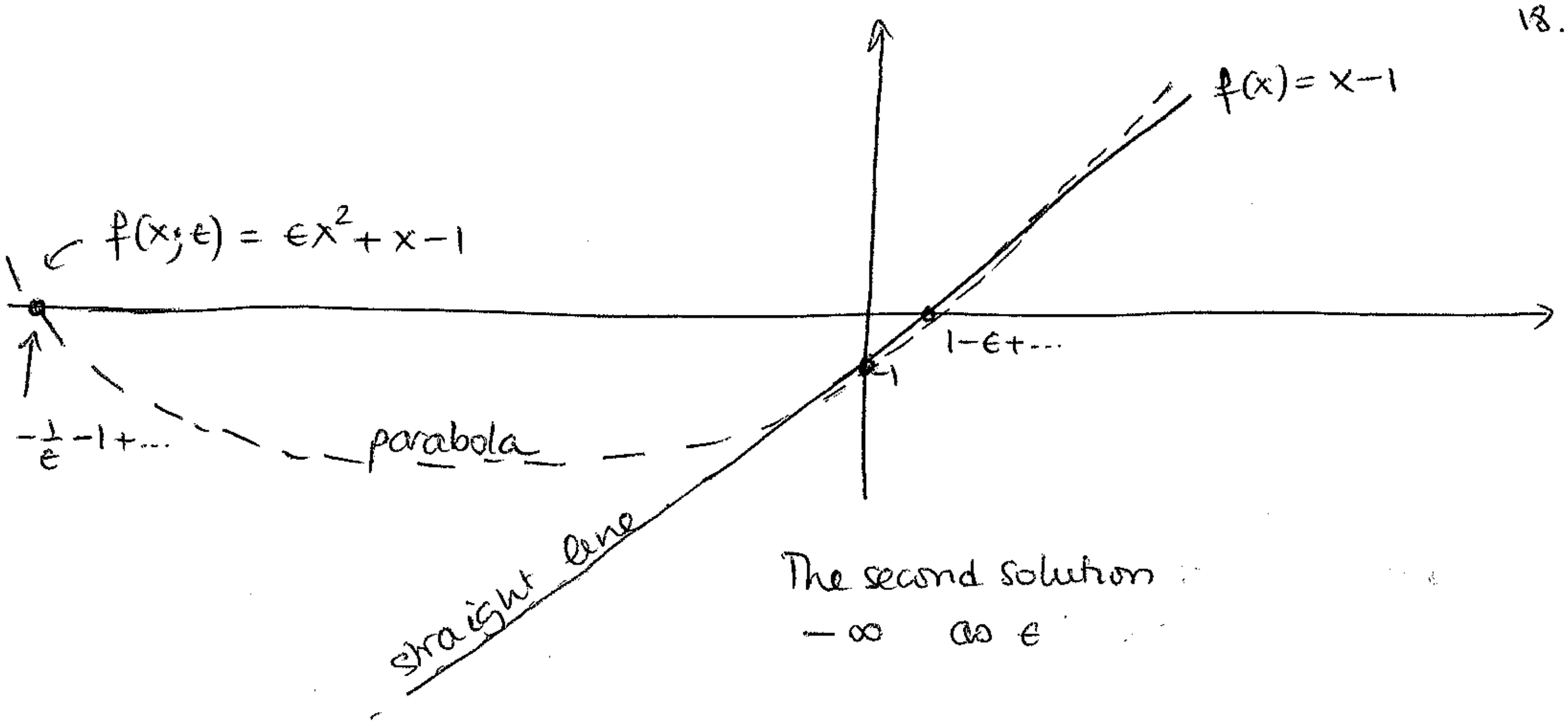
In this case both solutions are regular expansions in ϵ, and can be obtained from the same ansatz.

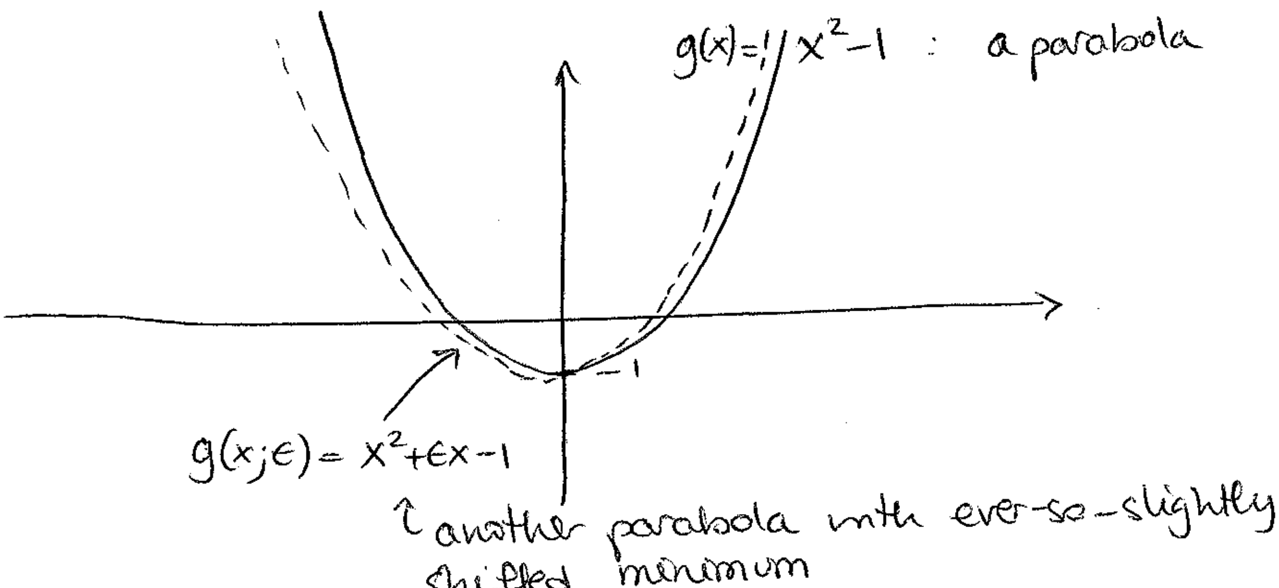
e. Graphical interpretation

To understand the origin of the "singular" solution in the case $\epsilon x^2 + x - 1 = 0$, but its absence in the case $x^2 + \epsilon x - 1 = 0$, consider the graphs associated with the functions

$$f(x) = x - 1 \qquad f(x;\epsilon) = \epsilon x^2 + x - 1$$

$$g(x) = x^2 - 1 \qquad g(x;\epsilon) = x^2 + \epsilon x - 1$$





$\rightarrow$ on the one hand there is a structural change in the graph of $f(x)$ as $\epsilon \neq 0$, on the other hand the graph of $g(x)$ is merely shifted by tiny amount when $\epsilon > 0$.

$\Rightarrow$ A singular solution can be expected when setting $\epsilon = 0$ in the equation completely changes its nature.

Otherwise, we can expect that there will only be regular solutions

This statement is quite general.

For instance we can expect singular solutions

- in ODEs / in PDEs where ϵ multiplies highest-order derivative
- in polynomials " " " highest-order term

⋮

etc.

f. Back to the original problem: singular ansatz

If, in the equation $\epsilon x^2 + x - 1 = 0$ we now assume a singular expansion of the form

$$x = \frac{s_{-1}}{\epsilon} + s_0 + \epsilon s_1 + \epsilon^2 s_2 + \cdots$$

then: $\epsilon\left(\frac{s_{-1}}{\epsilon} + s_0 + \epsilon s_1 + \cdots\right)^2 + \left(\frac{s_{-1}}{\epsilon} + s_0 + \epsilon s_1 + \cdots\right) - 1 = 0$

$\rightarrow$ to $O(\epsilon^{-1})$: $s_{-1}^2 + s_{-1} = 0 \Rightarrow s_{-1} = 0$ or $s_{-1} = -1$

this will give the regular solution — This will give the singular solution

for the singular one:

to $O(\epsilon^0)$: $2s_{-1}s_0 + s_0 - 1 = 0$

$$s_0 = \frac{1}{2s_{-1}+1} = \frac{1}{-1} = -1$$

to $O(\epsilon)$: $s_0^2 + 2s_{-1}s_1 + s_1 = 0$

$$\Rightarrow \quad s_1 = \frac{-s_0^2}{2s_{-1}+1} = \frac{-1}{-1} = 1$$

etc ...

So we get $x = -\frac{1}{\epsilon} - 1 + \epsilon + \cdots$

as expected. ✓

9. Another not-so-obvious example

Now consider instead the equation

$$x^2 - (2+\epsilon)x + 1 = 0$$

Let's try this time to naively assume that

$$x = s_0 + \epsilon s_1 + \epsilon^2 s_2 + \cdots$$

→ since ϵ does not multiply the highest degree term, this should work, right?

Let's proceed:

$$(s_0 + \epsilon s_1 + \epsilon^2 s_2 ..)^2 - (2+\epsilon)(s_0 + \epsilon s_1 + \epsilon^2 s_2 + ..) + 1 = 0$$

to $O(\epsilon^0)$: $s_0^2 - 2s_0 + 1 = 0 \rightarrow (s_0 - 1)^2 = 0$

$\Rightarrow s_0 = 1$ (double root)

to $O(\epsilon)$:

$$2s_0 s_1 - s_0 - 2s_1 = 0$$

$$\cancel{2s_1} - \cancel{2s_1} - 1 = 0 \Rightarrow -1 = 0 \ ?!$$

This seems to mean there is no solution.

What's happening?

To see our way through the problem, let's try instead the iterative method.

To lowest order: $[x^{(0)}]^2 - 2x^{(0)} + 1 = 0$

$\Rightarrow (x^{(0)} - 1)^2 = 0 \Rightarrow x^{(0)} = 1$

Then $[x^{(1)}]^2 - 2x^{(1)} + 1 - \epsilon x^{(0)} = 0$

$\Rightarrow (x^{(1)} - 1)^2 = \epsilon$

$\Rightarrow x^{(1)} = 1 \pm \sqrt{\epsilon}$ ← aha. here we see that the asymptotic sequence has $\sqrt{\epsilon}$

terms $\rightarrow$ now we can try instead the ansatz

$$x = S_0 + \sqrt{\epsilon}\, S_1 + \epsilon S_2 + \epsilon^{3/2} S_3 + \cdots$$

$$(S_0 + \sqrt{\epsilon}\, S_1 + \epsilon S_2 + \cdots)^2 - (2+\epsilon)(S_0 + \sqrt{\epsilon}\, S_1 + \epsilon S_2 + \cdots) + 1 = 0$$

to $O(\epsilon^0)$: $S_0^2 - 2S_0 + 1 = 0$

$\rightarrow S_0 = 1$

to $O(\epsilon^{1/2})$: $2S_0S_1 - 2S_1 = 0 \Rightarrow 2S_1 - 2S_1 = 0$

$\rightarrow$ This is consistent, but doesn't tell us what S_1 is ...

to $O(\epsilon)$: $S_1^2 + 2S_0S_2 - S_0 - 2S_2 = 0$

$\Rightarrow S_1^2 + \cancel{2S_2} - 1 - \cancel{2S_2} = 0$

$\Rightarrow S_1^2 = 1 \Rightarrow S_1 = \pm 1$

let's follow the + root:

to $O(\epsilon^{3/2})$: $2S_1S_2 + \cancel{2S_0S_3} - S_1 - \cancel{2S_3} = 0$

$\Rightarrow S_1(2S_2 - 1) = 0 \qquad S_2 = \frac{1}{2}$

In fact, we see that the same will be true for the − root.

So, to $O(\epsilon)$ we have

$$x = 1 \pm \sqrt{\epsilon} + \frac{\epsilon}{2} + \cdots$$

let's check this against the "true" answer

$$x = \frac{2+\epsilon \pm \sqrt{(2+\epsilon)^2 - 4}}{2}$$

$$= \frac{2+\epsilon \pm \sqrt{4\epsilon + \epsilon^2}}{2} = 1 + \frac{\epsilon}{2} \pm \sqrt{\epsilon}\,\frac{\sqrt{4+\epsilon}}{2}$$

$$= 1 + \frac{\epsilon}{2} \pm \sqrt{\epsilon}\left(1 + \frac{\epsilon}{8} + \cdots\right) \quad \checkmark$$

In this case, although the expansion wasn't singular, it wasn't obvious either – the reason being that we perturbed an equation that has a single (degenerate) root into one that has two.

See textbook for more examples of polynomial equations as well as practice problems.

Summary of II.1

→ In order to solve polynomial equations.

- determine whether equation may have singular solution or not
- if not, start with iterative method to get the power of ϵ in the asymptotic sequence, then use ansatz for asymptotic sequence to get actual approximate solution.
- if yes, then be aware that singular solution likely contains terms in $\frac{1}{(\epsilon)^\alpha}$ → Try ansatz of that kind (see more on this later too).