CHAPTER 7: Asymptotic expansions of integrals

In this final chapter, we focus on finding approximate values of integrals of the kind

$$\int_a^b e^{\lambda h(t)} f(t)\,dt \quad , \int_a^b \cos(\lambda h(t))\, f(t)\,dt \quad \text{and}$$

$$\int_a^b \sin(\lambda h(t))\, f(t)\,dt$$ in the limit where λ is large and real.

Note that this can be viewed in a unified way as all integrals of the kind $\int_a^b e^{\lambda h(t)} f(t)\,dt$ where λ is complex, and $|\lambda|$ is large.

These integrals are often called Laplace integrals in view of their relationship to Laplace Transforms, in which $h(t) = -t$. Indeed, the Laplace transform of the function $f(t)$ is (s complex) $\tilde{f}(s)$ where

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt .$$

We are then often interested in the behavior of $\tilde{f}(s)$ as $s \to \infty$

They also often arise in 1st order ODEs of the kind

$$\frac{dy}{dt} + \lambda p(t)\, y = f(t)$$

$$\Rightarrow \quad \mu(t) = e^{\lambda \int p(t')\,dt'} \quad \text{and so}$$

$$\frac{d}{dt}\left(y e^{\lambda \int p(t')\,dt'}\right) = \int f(t)\, e^{\lambda \int p(t')\,dt'}\,dt$$

$$= \int f(t)\, e^{\lambda h(t)}\,dt$$

if $h(t) = \int p(t')\,dt'$

finally, another common example is in the search for the steady state of an advection diffusion equation with source term: if

$$\frac{\partial u}{\partial t} + \underbrace{V(x)}_{\text{known velocity field}} \frac{\partial u}{\partial x} = \underbrace{\epsilon}_{\text{small diffusivity}} \frac{\partial^2 u}{\partial x^2} + \underbrace{S(x)}_{\text{source}}$$

Then we want to solve

$$-\epsilon \frac{\partial^2 u}{\partial x^2} + V(x)\frac{\partial u}{\partial x} = S(x)$$

Although we learned some methods from asymptotic theory to solve this directly, note that we can also write this as

$$-\epsilon \frac{\partial}{\partial x}\left(e^{-\frac{\int V(x')dx'}{\epsilon}} \frac{\partial u}{\partial x}\right) = e^{-\frac{\int V(x')dx'}{\epsilon}} S(x)$$

So the solution is formally given by

$$e^{-\frac{\int V(x')dx'}{\epsilon}} \frac{\partial u}{\partial x} = -\underbrace{\int \frac{1}{\epsilon}\, e^{-\frac{1}{\epsilon}\int V(x')\,dx'} S(x)\,dx}_{\text{again of the form } \int e^{\lambda h(x)} S(x)\,dx.}$$

Many other examples exist –

- in probability theory
- inverse Fourier transforms
- ....

I The region of dominant contribution

Reasonably simple approximations to integrals of the form $\int e^{-\lambda t} f(t)\, dt$ can be obtained by expanding $f(t)$ around the point in the interval considered where $e^{-\lambda t}$ is largest & make use of the fact that for large λ, $e^{-\lambda t}$ varies very rapidly.

Without loss of generality, we can take $\lambda > 0$ (if $\lambda < 0$, simply change variables $t \to -t$ in the integral).

Let's first start with the simple case where the integral is from 0 to ∞, and $f(t)$ is infinitely differentiable. The position where $e^{-\lambda t}$ is largest is clearly $t=0$ → Let's expand $f(t)$ around $t=0$: $f(t) = f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \cdots$

then,

$$\int_0^\infty e^{-\lambda t} f(t)\, dt \simeq \int_0^\infty e^{-\lambda t} \left(f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \cdots \right) dt$$

$$= \sum_{n=0}^{\infty} \int_0^\infty e^{-\lambda t} f^{(n)}(0) \frac{t^n}{n!}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \underbrace{\int_0^\infty e^{-\lambda t} t^n\, dt}$$

this is in fact equal to $\frac{\Gamma(n+1)}{\lambda^{n+1}}$ (see below), also equal to $\frac{n!}{\lambda^{n+1}}$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\lambda^{n+1}}$$

For large enough λ, this sum usually converges

Example: $\int_0^\infty e^{-\lambda t} \cos t \, dt = ?$

$$\cos^{(n)}(0) = (-1)^n \Rightarrow \int_0^\infty e^{-\lambda t} \cos t \, dt = \sum_n \frac{(-1)^n}{\lambda^{n+1}} = \frac{1}{\lambda} - \frac{1}{\lambda^3} + \frac{1}{\lambda^5} - \frac{1}{\lambda^7} \cdots$$

As it so happens, this integral can also be evaluated analytically, and is equal to $\frac{1}{\lambda(1+\frac{1}{\lambda^2})} = \frac{1}{\lambda}(1 - \lambda^2 + \lambda^4 + \cdots)$

$$= \frac{1}{\lambda} - \frac{1}{\lambda^3} + \frac{1}{\lambda^5} \cdots \checkmark$$

Let's now look at the slightly more complicated case of the integral $I = \int_a^b e^{-\lambda t} f(t) \, dt$.

We can always shift the lower bound to 0 by a simple change of variables $u = t - a$:

$$I = \int_0^{b-a} e^{-\lambda(u+a)} f(u+a) \, du$$

$$= e^{-\lambda a} \int_0^{b-a} e^{-\lambda u} g(u) \, du \qquad \text{where } g(u) = f(u+a)$$

The trick is to express the remaining integral as

$$\int_0^{b-a} e^{-\lambda u} g(u) \, du = \underbrace{\int_0^\infty e^{-\lambda u} g(u) \, du}_{\text{this we know how to deal with.}} - \int_{b-a}^\infty e^{-\lambda u} g(u) \, du$$

Then finally change variables again to make the lower bound of the second integral be 0: $v = u - (b-a)$:

$$\int_{b-a}^\infty e^{-\lambda u} g(u) \, du = \int_0^\infty e^{-\lambda(v+b-a)} g(v+b-a) \, dv$$

$$= e^{-\lambda(b-a)} \int_0^\infty e^{-\lambda v} g(v+b-a) \, dv.$$

What we have effectively done is to rewrite the original integral as

$$I = e^{-\lambda a}\int_0^\infty e^{-\lambda u} f(u+a)\,du - e^{-\lambda b}\int_0^\infty e^{-\lambda v} f(v+b)\,dv$$

since $\frac{d}{dx}\left(f(x+a)\right) = \left.\frac{df}{dx}\right|_{x+a}$ then

$$I = e^{-\lambda a}\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{\lambda^{n+1}} \quad e^{-\lambda b}\sum_{n=0}^{\infty}\frac{f^{(n)}(b)}{\lambda^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)e^{-\lambda a} - f^{(n)}(b)e^{-\lambda b}}{\lambda^{n+1}}$$

since λ is large, the terms in $e^{-\lambda b}$ will in general be $\ll$ those in $e^{-\lambda a}$ $\rightarrow$

$$I \cong \sum_{n=0}^{\infty}\frac{f^{(n)}(a)e^{-\lambda a}}{\lambda^{n+1}}$$

<u>Example</u>: $\int_1^3 e^{-\lambda t}\frac{1}{\sqrt{1+t^2}}\,dt \simeq \sum_{n=0}^{\infty} e^{-\lambda}\cdot\frac{f^{(n)}(1)}{\lambda^{n+1}}$

where $f(t) = \frac{1}{\sqrt{1+t^2}}$. To find the $f^{(n)}(1)$, it's actually easier to expand $f(t)$ near $t=1$:

$$f(1+\epsilon) = \frac{1}{\sqrt{1+(1+\epsilon)^2}} = \frac{1}{\sqrt{2+2\epsilon+\epsilon^2}} = \frac{1}{\sqrt{2}\sqrt{1+\epsilon+\frac{\epsilon^2}{2}}}$$

$$= \frac{1}{\sqrt{2}}\left(1-\left(\epsilon+\frac{\epsilon^2}{2}\right)\cdot\frac{1}{2} + \frac{3}{8}\left(\epsilon+\frac{\epsilon^2}{2}\right)^2 + \cdots\right)$$

$$= \frac{1}{\sqrt{2}}\left(1-\frac{\epsilon}{2}-\frac{\epsilon^2}{4}+\frac{3}{8}\epsilon^2+\frac{3}{8}\epsilon^3+\frac{3}{8}\cdot\frac{\epsilon^4}{4}+\cdots\right)$$

$$= \frac{1}{\sqrt{2}} + \epsilon f'(1) + \frac{\epsilon^2}{2}f''(1) + \frac{\epsilon^3}{6}f^{(3)}(1) + \cdots$$

$$\rightarrow\quad f'(1) = -\frac{1}{2\sqrt{2}} \qquad f''(1) = \frac{2}{\sqrt{2}}\left(\frac{3}{8}-\frac{1}{4}\right) = \frac{2}{\sqrt{2}}\cdot\frac{1}{8} = \frac{1}{4\sqrt{2}} \cdots$$

$$\Rightarrow \int_1^3 e^{-\lambda t}\frac{1}{\sqrt{1+t^2}}\,dt = e^{-\lambda}\left\{\frac{1}{\lambda\sqrt{2}} - \frac{1}{2\lambda^2\sqrt{2}} + \frac{1}{4\lambda^3\sqrt{2}}\cdots\right\}.$$

II Watson's Lemma & extensions

We have now all the tools to prove / understand

① Watson's Lemma

Let $I(\lambda) = \int_0^T e^{-\lambda t} t^a g(t)\,dt \qquad (T > 0)$

- where $a > -1$
- where $g(t)$ e is exponentially bounded in $[0,T]$
- where $g(t)$ has the McLaurin series $\sum_{n=0}^{\infty} g^{(n)}(0) \frac{t^n}{n!}$

then $I(\lambda) \sum_{n=0}^{\infty} g^{(n)}(0) \frac{\Gamma(a+n+1)}{n!\, \lambda^{a+n+1}}$ as $\lambda \to +\infty$.

The proof is very similar to the steps we have just followed to calculate $\int_a^b e^{-\lambda t} f(t)\,dt$.

Note : (i) The condition $a > -1$ effectively guarantees that the lower bound of the integral $I(\lambda)$ exists

(i.e. $\lim_{\epsilon \to 0} \left| \int_\epsilon^T e^{-\lambda t} t^a g(t)\,dt \right| < +\infty$)

(ii) If $g(0) = 0$, or $g(0)$ and $g'(0) = 0$, etc, this constraint can be relaxed (it is sufficient that the first term in the McLaurin series times t^a be greater than $\frac{1}{t}$ as $t \to 0$).

(iii) $g(t)$ being exponentially bounded means that $|g(t) e^{-\lambda t}| < K$ for some K → the decay of the $e^{-\lambda t}$ term must dominate the behavior of $g(t)$

(iv) If T is not in the radius of convergence of the McLaurin series, then the proof needs to be more careful but the result still holds (see book)

② Generalizations

We can now generalize the results to the case initially listed: $I = \int_a^b e^{-\lambda h(t)} f(t)\, dt$.

To do so, simply let $u = h(t)$ (assuming $h(t)$ is one-to-one)

Then $$I = \int_{h(a)}^{h(b)} e^{-\lambda u} f(h^{-1}(u)) \frac{du}{h'(h^{-1}(u))} = \int_{h(a)}^{h(b)} e^{-\lambda u} g(u)\, du$$

and we can, once more, apply the same technique.

Example 1: $I = \int_0^\infty e^{-\lambda t^2} \sin(t)\, dt$

let $u = t^2$ (which is one-to-one when $t \geq 0$)

then $$I = \int_0^\infty \sin\sqrt{u}\, e^{-\lambda u} \frac{du}{2t} = \int_0^\infty \sin\sqrt{u}\, e^{-\lambda u} \frac{du}{2\sqrt{u}}$$

by Watson's lemma $$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\, \cancel{\Gamma(n+1)}}{\cancel{n!}\, \lambda^{n+1}}$$

($a = 0$ here)

Note that choosing $a = -\frac{1}{2}$ and $g(u) = \sin\sqrt{u}$ is not a good idea

where $g(u) = \frac{\sin\sqrt{u}}{\sqrt{u}}$

Again it's easier to get the derivatives of $g(u)$ by expanding $\sin\sqrt{u}$ as $\sin\sqrt{u} \simeq \sqrt{u} - \left(\frac{\sqrt{u}^3}{3!}\right) + \frac{(\sqrt{u})^5}{5!} - \cdots$

so $g(u) = 1 - \frac{u}{3!} + \frac{u^2}{5!} - \cdots$

so $g^{(0)}(0) = 1 \quad g^{(1)}(0) = -\frac{1}{3!} \quad g^{(2)}(0) = \frac{2}{5!} \cdots$

$$\Rightarrow I = \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{6\lambda^2} + \frac{1}{60\lambda^3} \cdots\right)$$

Example 2: $\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_\lambda^\infty e^{-t^2}\, dt$

This one is a bit odd since λ appears in the bound instead of the integral. However, with $u = t - \lambda$ then

$$\mathrm{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(u+\lambda)^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - 2\lambda u - \lambda^2} du$$

$$= \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \int_0^\infty e^{-u^2} e^{-2\lambda u} du$$

$$= \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \int_0^\infty \underbrace{e^{-u^2}}_{g(u)} e^{-\tilde{\lambda} u} du$$

$$= \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{\tilde{\lambda}^{n+1}}$$ where $g^{(n)}(0)$ can be obtained by Taylor expanding e^{-u^2} near $u=0$:

$$e^{-u^2} = 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots$$

$$\rightarrow g^{(0)}(0) = 1 \quad g^{(1)}(0) = 0 \quad g^{(2)}(0) = -2$$
$$g^{(3)}(0) = 0 \quad g^{(4)}(0) = \frac{4!}{2!} \quad \text{etc...}$$

$$\Rightarrow \mathrm{erfc}(\lambda) \approx \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \left(\frac{1}{\tilde{\lambda}} - \frac{2}{\tilde{\lambda}^3} + \frac{12}{\tilde{\lambda}^5} \cdots \right)$$
$$\sim \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \left(\frac{1}{2\lambda} - \frac{1}{4\lambda^3} + \frac{3}{8\lambda^5} \cdots \right)$$