V Example of application: stellar oscillations (part 1)

The surface of stars oscillates as a result of global pressure (sound) waves that are excited in convective regions and travel throughout the interior. The pressure field within the star satisfies a wave equation of the kind $\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2(r)\nabla^2 \tilde{p}$ where $\tilde{p}$ is the deviation from hydrostatic equilibrium, c is the local sound speed (assumed here to depend only on the radial coordinate r) and

$$\nabla^2 \tilde{p} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{p}}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \tilde{p}}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \tilde{p}}{\partial \phi^2}$$

In what follows, we study axisymmetric waves, where $\frac{\partial}{\partial \phi} = 0$. We assume that $\tilde{p}$ must be 0 at the stellar surface (for simplicity) and must be regular at the center.

① Separation of variables

- First separation of variables suggests that $\tilde{p}$ oscillates with time, so that $\tilde{p}(r,\theta,t) = A(r,\theta)B(t)$ with $\frac{d^2 B}{dt^2} = -\omega^2 B$ so that A satisfies

$$\nabla^2 A = -\frac{\omega^2}{c^2(r)} A$$

ω is an eigenvalue of the problem, determined by fitting A through the BCs.

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A}{\partial \theta}\right) = -\frac{\omega^2}{c^2(r)} A .$$

- Second separation of variables $A(r,\theta) = f(r)\,g(\theta)$

leaves:
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial g}{\partial\theta}\right) = \text{const.}\; g$$

This is an equation for a Legendre function $P_\ell(\cos\theta)$, whose eigenvalue is $-\ell(\ell+1)$ for solutions to be regular at $\theta = \pm\frac{\pi}{2}$ $\Rightarrow$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dg}{d\theta}\right) = -\,\ell(\ell+1)\,g$$ where ℓ is an integer.

This finally leaves the radial equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{\ell(\ell+1)}{r^2}f = -\frac{\omega^2}{c^2(r)}f$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \left(\frac{\omega^2}{c^2(r)} - \frac{\ell(\ell+1)}{r^2}\right)f = 0$$

② Removal of 1st derivative:

$$\Rightarrow \quad \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \left(\frac{\omega^2}{c^2(r)} - \frac{\ell(\ell+1)}{r^2}\right)f = 0$$

So let $y(r) = f(r)\,e^{\frac{1}{2}\int\frac{2}{r'}dr'} = r\,f(r)$

$$\Rightarrow f(r) = \frac{y(r)}{r}$$

$$\frac{df}{dr} = \frac{1}{r}\frac{dy}{dr} - \frac{y}{r^2} \qquad \frac{d^2f}{dr^2} = \frac{1}{r}\frac{d^2y}{dr^2} - \frac{2}{r^2}\frac{dy}{dr} + \frac{2y}{r^3}$$

$$\Rightarrow \frac{1}{r}\frac{d^2y}{dr^2} - \cancel{\frac{2}{r^2}\frac{dy}{dr}} + \cancel{\frac{2y}{r^3}} + \frac{2}{r}\left(\cancel{\frac{1}{r}\frac{dy}{dr}} - \cancel{\frac{y}{r^2}}\right)$$
$$+ \left[\frac{\omega^2}{c^2(r)} - \frac{\ell(\ell+1)}{r^2}\right]\frac{y}{r} = 0$$

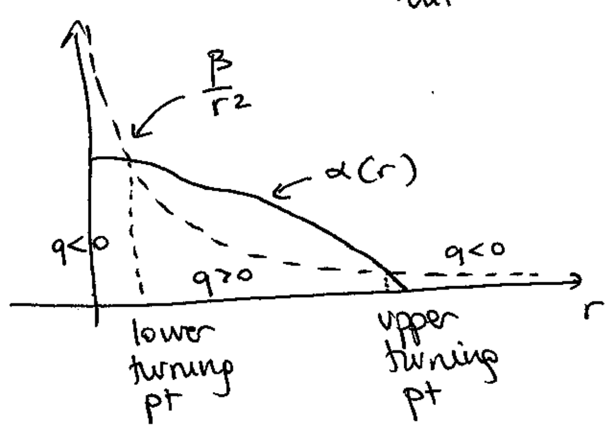
$$\Rightarrow \quad \frac{d^2y}{dr^2} + \underbrace{\left[\frac{\omega^2}{c^2(r)} - \frac{\ell(\ell+1)}{r^2}\right]}_{q(r;\,\ell,\,\omega)}y = 0$$

ℓ: integer, known; ω: eigenfrequency, TBD

③ Sign of $q(r)$

- Let's assume for the moment that $c^2(r)$ is constant. In that case $q(r)$ is of the form $\alpha - \frac{\beta}{r^2}$ where α and β are both positive. This vanishes at the point $r_{crit} = \sqrt{\frac{\beta}{\alpha}}$. If this point is within the stellar interior, then it is a turning point. For $r < r_{crit}$ $q(r) < 0$ and for $r > r_{crit}$, $q(r) > 0$.

- In general, c is not constant, but instead decreases with r; so we have $q(r)$ of the form $\alpha(r) - \frac{\beta}{r^2}$. However since α is bounded as $r \to 0$, while $\frac{\beta}{r^2}$ isn't, we also usually have a turning point at r_{crit} such that $\alpha(r_{crit}) = \frac{\beta}{r_{crit}^2}$. Interestingly, it is also possible to have 2 turning pts; eg when $\alpha(r)$ is convex when $\frac{\beta}{r^2}$ is concave.



⇒ In this problem, we have to deal with 2 of the difficulties we have been putting off so far. Eigenvalues & turning pts. Let's now study them in turn.

VI Eigenproblems in the WKB approximation (case with no turning pt)

If there is no turning pt, then we saw that the general solution of the problem

$$\frac{d^2 y}{dx^2} + \lambda^2 q(x) y = 0$$

, in the limit of large $\lambda > 0$, is

$$y(x) = \frac{1}{|q(x)|^{1/4}} \left\{ a e^{i\lambda \int \sqrt{q(x')}\, dx'} + b e^{-i\lambda \int \sqrt{q(x')}\, dx'} \right\}$$

Given some boundary conditions, this problem could be viewed as an eigenvalue problem for λ. In fact, this is a Sturm-Liouville problem which is regular (given appropriate BCs), and so we know that the eigenvalues λ form an infinite sequence that is unbounded from above: $\lambda_0 < \lambda_1 < \dots \lambda_n < +\infty$

→ $\boxed{\text{The WKB approximation can be used to give us the eigenvalues \& eigenmodes in the limit of large } \lambda.}$

Example 1: Find approximate eigenvalues & eigenmodes of the eigenvalue problem $\frac{d^2y}{dx^2} + \lambda^2 (1+x^2)^2 y = 0$ with $y(0) = 0$ and $y(1) = 0$ in the $\lambda \to \infty$ limit

→ We saw that the solution is (in the limit of large λ)

$$y(x) \cong \frac{1}{\sqrt{1+x^2}} \left\{ a \cos\left(\lambda\left(x + \frac{x^3}{3}\right)\right) + b \sin\left(\lambda\left(x + \frac{x^3}{3}\right)\right) \right\}$$

$y(0) = 0 \Rightarrow a = 0$

$y(1) = 0 \Rightarrow \frac{1}{\sqrt{2}} b \sin\left(\frac{4}{3}\lambda\right) = 0 \Rightarrow \frac{4}{3}\lambda = n\pi \Rightarrow \lambda_n = \frac{3n\pi}{4}$

so $\begin{cases} \lambda_n = \dfrac{3n\pi}{4} \\ y_n(x) = \dfrac{1}{\sqrt{1+x^2}} \sin\left(\dfrac{3n\pi}{4}\left(x + \dfrac{x^3}{3}\right)\right) \end{cases}$ are the eigenvalues & eigenmodes in the limit of large n (large λ)

<u>Example 2</u>: find the eigenvalues & eigenmodes of $\dfrac{d^2y}{dx^2} - \dfrac{\lambda y}{x} =$ in the $\lambda \to \infty$ limit, where BCs are

$$y(-2) = 0 \quad \frac{dy}{dx}(-1) = 0 .$$

$$y(x) = |x|^{1/4}\left(a\cos\left(2\sqrt{\lambda|x|}\right) + b\sin\left(2\sqrt{\lambda|x|}\right)\right)$$

<u>homework: check this</u>

$y(-2) = 0 \Rightarrow$

$$\cancel{2^{1/4}}\left(a\cos\left(2\sqrt{2\lambda}\right) + b\sin\left(2\sqrt{2\lambda}\right)\right) = 0$$

$$\Rightarrow \quad a = -\,b\tan\left(2\sqrt{2\lambda}\right)$$

$$\frac{dy}{dx} = \frac{dy}{d|x|}\cdot\frac{d|x|}{dx} = -\frac{dy}{d|x|} \quad \text{when } x<0$$

$$= -\left[\frac{1}{4}|x|^{-3/4}\left(a\cos\left(2\sqrt{\lambda|x|}\right) + b\sin\left(2\sqrt{\lambda|x|}\right)\right)\right.$$
$$\left. + |x|^{1/4}\left(-a\sin\left(2\sqrt{\lambda|x|}\right)\sqrt{\frac{\lambda}{|x|}} + b\cos\left(2\sqrt{\lambda|x|}\right)\sqrt{\frac{\lambda}{|x|}}\right.\right.$$

so $\dfrac{dy}{dx}(-1) = 0 \Rightarrow$

$$\frac{1}{4}\left(a\cos 2\sqrt{\lambda} + b\sin 2\sqrt{\lambda}\right)$$
$$+\sqrt{\lambda}\left(-a\sin\left(2\sqrt{\lambda}\right) + b\cos\left(2\sqrt{\lambda}\right)\right) = 0$$

$$\Rightarrow \quad \left(\frac{b}{4} - a\sqrt{\lambda}\right)\sin\left(2\sqrt{\lambda}\right) + \left(\frac{a}{4} + b\sqrt{\lambda}\right)\cos\left(2\sqrt{\lambda}\right) = 0$$

$\Rightarrow$ λ satisfies the transcendental equation

$$\left(\frac{1}{4} + \sqrt{\lambda}\tan\left(2\sqrt{2\lambda}\right)\right)\sin\left(2\sqrt{\lambda}\right)$$
$$= -\left(\sqrt{\lambda} - \frac{1}{4}\tan\left(2\sqrt{2\lambda}\right)\right)\cos\left(2\sqrt{\lambda}\right)$$

$$\Rightarrow \tan\left(2\sqrt{\lambda}\right) = \frac{\frac{1}{4}\tan\left(2\sqrt{2\lambda}\right) - \sqrt{\lambda}}{\sqrt{\lambda}\tan\left(2\sqrt{2\lambda}\right) + \frac{1}{4}}$$

$\rightarrow$ $\left\{\begin{array}{l}\text{from here on,} \\ \text{must be solved} \\ \text{numerically.}\end{array}\right.$

VII Turning points in the WKB approximation

We now consider the case where $\frac{d^2y}{dx^2} + \lambda^2 q(x) y = 0$, where $q(x)$ has a zero in the interval of interest. As we saw earlier, this point is called a "turning pt".

In order to solve this problem, the idea is to solve for $y(x)$ on either side of the turning pt using WK[illegible] and then match the solutions on the two sides to create a complete solution valid for all x in the interval.

The matching procedure hangs crucially on the exact solutions of the Airy equation:

$$\frac{d^2 f}{d\xi^2} - f\xi = 0$$

$$\rightarrow \quad f(\xi) = \alpha \, Ai(\xi) + \beta \, Bi(\xi)$$

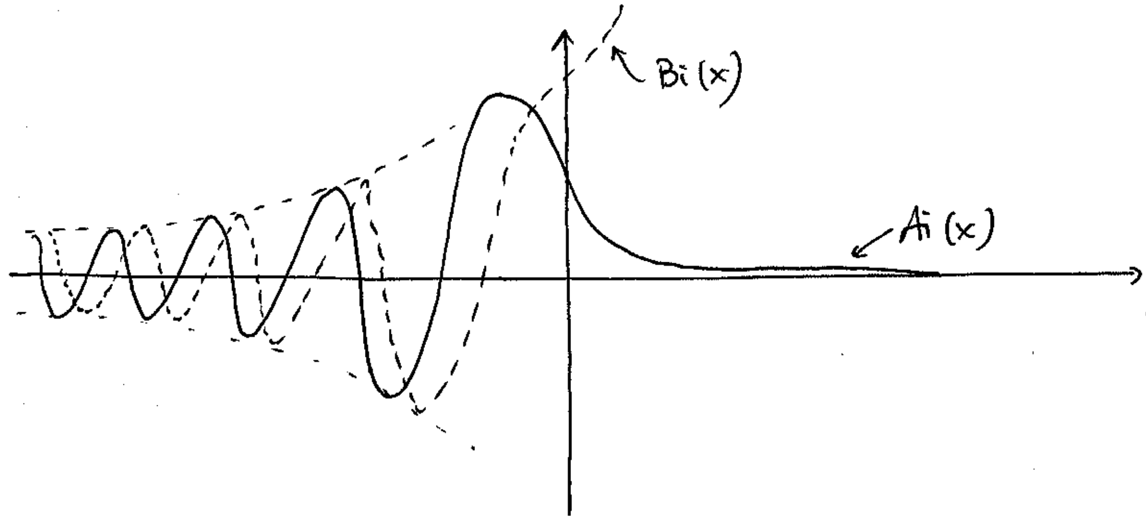
Properties of the Airy functions (see also Example 1 in III)

- $Ai(0) = \dfrac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}$ $\qquad Ai'(0) = -\dfrac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}$

  $Bi(0) = \dfrac{1}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})}$ $\qquad Bi'(0) = \dfrac{3^{1/6}}{\Gamma(\frac{1}{3})}$

- as $x \to +\infty$
  $$Ai(x) \simeq \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}}$$
  $$Bi(x) \simeq \frac{1}{\sqrt{\pi}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}}$$

- as $x \to -\infty$
  $$Ai(x) \simeq \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)$$
  $$Bi(x) \simeq \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4}} \cos\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)$$



The matching procedure otherwise proceeds exactly as if we were dealing with an internal boundary layer; the "outer" solutions are given by the WKB approximation far from the turning pt, & the "inner" solution is given by the Airy function. To see this, let's consider the example

$$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda^2(x-1)\,y = 0 \\ y(0) = 0 \\ y(2) = 0 \end{cases}$$

For x <u>not</u> close to 1, we have

- for $x<1$, $q(x) < 0$ so

$$y^{outer}(x) = \frac{1}{(1-x)^{1/4}} \left\{ A\exp\left(\lambda \int \sqrt{1-x'}\,dx'\right) + B\exp\left(-\lambda \int \sqrt{1-x'}\,dx'\right) \right\}$$

$$= \frac{1}{(1-x)^{1/4}} \left\{ A\exp\left(\tfrac{2}{3}\lambda(1-x)^{3/2}\right) + B\exp\left(-\tfrac{2}{3}\lambda(1-x)^{3/2}\right) \right\}.$$

- for $x>1$, $q(x)>0$ so

$$y^{outer}(x) = \frac{1}{(x-1)^{1/4}} \left\{ C\cos\left(\frac{2\lambda}{3}(x-1)^{3/2}\right) + D\sin\left(\frac{2\lambda}{3}(x-1)^{3/2}\right) \right\}$$

We now find the inner solution as usual.

let $s = \lambda^p (x-1) \rightarrow \frac{d}{dx} = \lambda^p \frac{d}{ds} \rightarrow \frac{d^2}{dx^2} = \lambda^{2p} \frac{d^2}{ds^2}$

and $x = 1 + \frac{s}{\lambda^p}$

$\Rightarrow \lambda^{2p} \frac{d^2 y}{ds^2} + \lambda^2 \frac{s}{\lambda^p} y = 0 \quad \Rightarrow$ matching orders we get

$2p = 2 - p$ so

$p = \frac{2}{3}$

To lowest order, $\frac{d^2 y}{ds^2} + ys = 0 \rightarrow y$ is an Airy function

$y^{inner}(s) = a\,Ai(-s) + b\,Bi(-s)$

Finally, we now match all of these solutions to one another and to the BCs:

BCs: $y(0) = 0 \Rightarrow \quad A \exp\left(\frac{2}{3}\lambda\right) + B\exp\left(-\frac{2}{3}\lambda\right) = 0$

$\Rightarrow \quad B = -Ae^{\frac{4}{3}\lambda}$

$y(2) = 0 \Rightarrow \quad C\cos\left(\frac{2\lambda}{3}\right) + D\sin\left(\frac{2\lambda}{3}\right) = 0$

$\Rightarrow \quad C = -D\tan\left(\frac{2\lambda}{3}\right)$

Prandtl's matching condition on left of BL:

$\lim_{x \to 1_-} y^{outer} = \frac{1}{(1-x)^{1/4}} \left\{ A\exp\left(\frac{2}{3}\lambda(1-x)^{3/2}\right) + B\exp\left(-\frac{2}{3}\lambda(1-x)^{3/2}\right)\right\}$

$\rightarrow$ can't do much about it since it's singular.

however, let's look at inner:

$\lim_{s \to -\infty} y^{inner}(s) = \frac{a}{2\sqrt{\pi}} |s|^{-1/4} e^{-\frac{2}{3}|s|^{3/2}} + \frac{b}{\sqrt{\pi}} |s|^{-1/4} e^{\frac{2}{3}|s|^{3/2}}$

since $s = \lambda^{\frac{2}{3}}(x-1)$, for $x < 1$ $|s| = \lambda^{\frac{2}{3}}(1-x)$

so $|s|^{3/2} = \lambda(1-x)^{3/2}$

We can then identify, term by term,

$$\frac{A}{(1-x)^{1/4}} = \frac{b}{\sqrt{\pi}}\left[\lambda^{\frac{2}{3}}(1-x)\right]^{-\frac{1}{4}}$$

$$\Rightarrow \quad A = \frac{b\lambda^{-\frac{1}{6}}}{\sqrt{\pi}} = \frac{b}{\lambda^{\frac{1}{6}}\sqrt{\pi}}$$

and $$\frac{B}{(1-x)^{1/4}} = \frac{a}{2\sqrt{\pi}}\left[\lambda^{\frac{2}{3}}(1-x)\right]^{-\frac{1}{4}}$$

$$\Rightarrow \quad B = \frac{a}{2\sqrt{\pi}}\lambda^{-\frac{1}{6}}$$

Similarly, on the right side of BL:

$$\lim_{x\to 1_+} y^{outer} = \frac{1}{(x-1)^{1/4}}\left\{C\cos\left(\frac{2\lambda}{3}(x-1)^{3/2}\right) + D\sin\left(\frac{2\lambda}{3}(x-1)^{3/2}\right)\right\}$$

and $$\lim_{s\to+\infty} y^{inner}(s) = \frac{1}{s^{1/4}}\left\{\frac{a}{\sqrt{\pi}}\sin\left(\frac{2}{3}s^{\frac{3}{2}}+\frac{\pi}{4}\right) + \frac{b}{\sqrt{\pi}}\cos\left(\frac{2}{3}s^{3/2}+\frac{\pi}{4}\right)\right\}$$

$$= \frac{1}{s^{1/4}\sqrt{\pi}}\left\{a\sin\left(\frac{2}{3}s^{3/2}\right)\cos\frac{\pi}{4} + a\cos\left(\frac{2}{3}s^{3/2}\right)\sin\frac{\pi}{4} + b\cos\left(\frac{2}{3}s^{3/2}\right)\cos\frac{\pi}{4} - b\sin\left(\frac{2}{3}s^{3/2}\right)\sin\frac{\pi}{4}\right\}$$

$$= \frac{\sqrt{2}}{2s^{1/4}\sqrt{\pi}}\left\{(a-b)\sin\left(\frac{2}{3}s^{3/2}\right) + (a+b)\cos\left(\frac{2}{3}s^{3/2}\right)\right\}$$

Identifying the two we get; using $s = (x-1)\lambda^{2/3}$

$$\frac{1}{s^{1/4}}\frac{\sqrt{2}}{2\sqrt{\pi}}(a-b) = \frac{D}{(x-1)^{1/4}} \Rightarrow \frac{1}{\lambda^{1/6}}\frac{\sqrt{2}}{2\sqrt{\pi}}(a-b) = D.$$

$$\frac{1}{s^{1/4}}\frac{\sqrt{2}}{2\sqrt{\pi}}(a+b) = \frac{C}{(x-1)^{1/4}} \Rightarrow \frac{1}{\lambda^{1/6}}\frac{\sqrt{2}}{2\sqrt{\pi}}(a+b) = C.$$

So finally:

- From $B = -Ae^{\frac{4}{3}\lambda}$ and $A = \frac{b}{\lambda^{1/6}\sqrt{\pi}}$, $B = \frac{a}{2\sqrt{\pi}\,\lambda^{1/6}}$

$$\Rightarrow \quad \frac{a}{2\cancel{\sqrt{\pi}}\cancel{\lambda^{1/6}}} = -\frac{e^{\frac{4}{3}\lambda}\,b}{\cancel{\lambda^{1/6}}\cancel{\sqrt{\pi}}} \Rightarrow \quad a = -2e^{\frac{4}{3}\lambda}\,b$$

- From $C = -D\tan\left(\frac{2\lambda}{3}\right)$ and equations above:

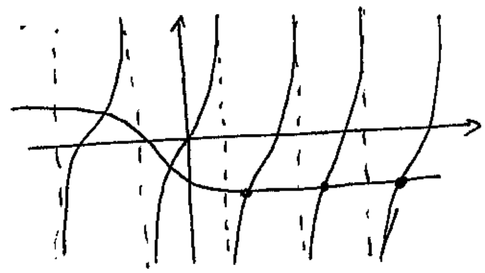
$$\Rightarrow \quad (a+b)\frac{\cancel{\sqrt{2}}}{\cancel{2\sqrt{\pi}}}\,\frac{1}{\cancel{\lambda^{1/6}}} = -\frac{(a-b)\cancel{\sqrt{2}}}{\cancel{\lambda^{1/6}}\,\cancel{2\sqrt{\pi}}}\tan\left(\frac{2\lambda}{3}\right)$$

$$\Rightarrow \quad a+b = -(a-b)\tan\left(\frac{2\lambda}{3}\right)$$

$$\Rightarrow \quad \cancel{b}\left(1-2e^{\frac{4}{3}\lambda}\right) = \cancel{b}\left(1+2e^{\frac{4}{3}\lambda}\right)\tan\left(\frac{2\lambda}{3}\right)$$

$$\Rightarrow \quad \frac{1-2e^{\frac{4}{3}\lambda}}{1+2e^{\frac{4}{3}\lambda}} = \tan\left(\frac{2\lambda}{3}\right)$$

$\Rightarrow$ this is a transcendental equation for the eigen value λ.



Note that if $u = \frac{2\lambda}{3}$, this is

$$\frac{1-2e^{2u}}{1+2e^{2u}} = \tan u$$

For large u this is approximately $\tan u = -1$

$$\Rightarrow \quad \boxed{u \simeq -\frac{\pi}{4} + n\pi} \quad \Rightarrow \quad \boxed{\lambda_n \simeq \frac{3}{2}u = -\frac{3\pi}{8} + \frac{3n\pi}{2}}$$

$\rightarrow$ we can thus find the large eigenvalues of the problem.

<u>Note</u>: there is one remaining undetermined constant. This is not surprising, since we were solving a linear eigen value problem.

<u>Note</u>: This entire problem could have been solved in 1 line... can you see how

In general, matching through a turning point can be mathematically fairly challenging, but the general steps taken here apply pretty much as is...

VIII. Stellar oscillations part 2

Let's now go back to our original problem & study for simplicity the case $c = c_0 =$ constant.
We will focus on the asymptotic limit of rapid oscillations, where ω is large.

In that case we can write

$$\frac{d^2 y}{dr^2} + \omega^2 \tilde{q}_\ell(r) y = 0 \quad \text{where} \quad \tilde{q}_\ell(r) = \frac{1}{c_0^2}\left(1 - \frac{r_{crit}^2}{r^2}\right)$$

$$\text{where} \quad r_{crit} = \frac{c_0 \ell}{\omega}$$

Clearly $\tilde{q}_\ell(r) < 0$ if $r < r_{crit}$ & $\tilde{q}_\ell(r) > 0$ if $r > r_{crit}$.

For $r < r_{crit}$

$$y(r) = \frac{1}{|\tilde{q}_\ell(r)|^{1/4}} \left\{ A \exp\left(\omega \int \sqrt{|\tilde{q}_\ell(r')|}\, dr'\right) + B \exp\left(-\omega \int \sqrt{|\tilde{q}_\ell(r')|}\, dr'\right) \right\}$$

$$= \frac{c_0^{1/2}}{\left(\frac{r_{crit}^2}{r^2} - 1\right)^{1/4}} \left\{ A \exp\left(\omega f(r)\right) + B \exp\left(-\omega f(r)\right) \right\}$$

$$\text{where} \quad f(r) = \frac{1}{c_0} \int \sqrt{\frac{r_{crit}^2}{r'^2} - 1}\, dr' = \frac{1}{2c_0}\left[r_{crit} \sqrt{\frac{r_{crit}^2}{r^2} - 1} - r \ln\left(r \sqrt{\frac{r_{crit}^2}{r^2} - 1} + r_{c}\right.\right.$$

For $r > r_{crit}$

$$y(r) = \frac{c_0^{1/2}}{\left(1 - \frac{r_{crit}^2}{r^2}\right)^{1/4}} \left\{ C \cos\left(\omega \int \sqrt{\tilde{q}_\ell(r')}\, dr'\right) + D \sin\left(\omega \int \sqrt{\tilde{q}_\ell(r')}\, dr'\right) \right\}$$

$$= \frac{c_0^{1/2}}{\left(1 - \frac{r_{crit}^2}{r^2}\right)^{1/4}} \left\{ C \cos\left(\omega g(r)\right) + D \sin\left(\omega g(r)\right) \right\}$$

$$\text{where} \quad g(r) = \frac{1}{c_0} \int \sqrt{1 - \frac{r_{crit}^2}{r'^2}}\, dr' = \frac{1}{2c_0}\left[r_{crit} \sqrt{1 - \frac{r_{crit}^2}{r^2}} + r \sin^{-1}\left(\frac{r_{crit}}{r}\right) \right]$$

Finally, we need the "innersolution" where $r \approx r_{crit}$.

let $s = (r - r_{crit})\omega^{p} \Rightarrow r = s\omega^{-p} + r_{crit}$

$$\frac{d}{dx} = \omega^{p}\frac{d}{ds} \Rightarrow \omega^{2p}\frac{d^2y}{ds^2} + \frac{\omega^2}{C_0^2}\left(1 - \frac{r_{crit}^2}{(r_{crit} + s\omega^{-p})^2}\right) y = 0$$

$$\Rightarrow \omega^{2p}\frac{d^2y}{ds^2} + \frac{\omega^2}{C_0^2}\left(1 - \frac{1}{\left(1 + \frac{s\omega^{-p}}{r_{crit}}\right)^2}\right) y = 0$$

$$\Rightarrow \omega^{2p}\frac{d^2y}{ds^2} + \frac{\omega^2}{C_0^2}\,\frac{2\omega^{-p}s}{r_{crit}}\, y = 0$$

$$\Rightarrow \omega^{2p} = \omega^{2-p} \Rightarrow p = 2/3$$

To lowest order:

$$\frac{d^2y}{ds^2} + \frac{2s}{C_0^2 r_{crit}}\, y = 0$$ → looks like an Airy equation, but not quite

We now want to write this as $\frac{d^2y}{d\xi^2} - \xi y = 0$.

let $\xi = \alpha s \Rightarrow \frac{d}{ds} = \alpha\frac{d}{d\xi} \Rightarrow$

$$\alpha^2\frac{d^2y}{d\xi^2} + \frac{2\xi}{\alpha C_0^2 r_{crit}}\, y = 0$$ so we need $\frac{2}{\alpha^3 C_0^2 r_{crit}} = -1$

$$\rightarrow \alpha = -\left(\frac{2}{C_0^2 r_{crit}}\right)^{1/3}.$$

If this is done, then the solution becomes:

$$y(\xi) = a\,Ai(\xi) + b\,Bi(\xi)$$

$$y(s) = a\,Ai\left(-\left(\frac{1}{C_0^2}\frac{2}{r_{crit}}\right)^{1/3} s\right) + b\,Bi\left(-\left(\frac{1}{C_0^2}\frac{2}{r_{crit}}\right)^{1/3} s\right)$$

$$y^{inner}(r) = a\,Ai\left((r_{crit} - r)\left(\frac{2}{C_0^2 r_{crit}}\right)^{1/3}\omega^{2/3}\right)$$

$$+ b\,Bi\left((r_{crit} - r)\left(\frac{2}{r_{crit}C_0^2}\right)^{1/3}\omega^{2/3}\right)$$ near $r = r_{crit}$

We now just have to apply BCs & matching conditions to get the frequencies. This part, however, is fairly ugly, but can be done analytically with a lot of algebra.