IV The WKB approximation

The WKB approximation is used to solve equations of the kind $\frac{d^2y}{dx^2} + q(x;\lambda)y = 0$ where λ is a <u>large</u> parameter, returning solutions valid for <u>all</u> x (at least in cases where the expansion found is uniform, which is not always necessarily the case, see later for an example).

To solve this equation essentially requires the method of dominant balance on the ansatz

$$y(x;\lambda) = \exp\left(\phi_0(x)\cdot\delta_0(\lambda) + \phi_1(x)\delta_1(\lambda) + \phi_2(x)\delta_2(\lambda) + \cdots\right)$$

where $\{\delta_n(\lambda)\}$ form an asymptotic sequence where $\delta_n(\lambda) = o(\delta_{n-1}(\lambda))$, and all the $\phi_n(x)$ are $O(1)$ in terms of λ (at fixed x)

Let's see how the method works on examples of the kind $\frac{d^2y}{dx^2} + \lambda^2 q(x)y = 0$, called "Liouville Equations".

As before, we have
$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} \phi_n''\delta_n(\lambda)\, e^{\sum\phi_n\delta_n} + \left(\sum_{n=0}^{\infty}\phi_n'\delta_n(\lambda)\right)^2 e^{\sum\phi_n\delta_n}$$

So, we have to solve:
$$\sum_{n=0}^{\infty}\phi_n''(x)\,\delta_n(\lambda) + \left(\sum_{n=0}^{\infty}\phi_n'(x)\delta_n(\lambda)\right)^2 + \lambda^2 q(x) = 0$$

Keeping the lowest orders in each term, we get
$$\phi_0''(x)\,\delta_0(\lambda) + \phi_0'^2(x)\,\delta_0^2(\lambda) + \lambda^2 q(x) = 0$$
Since $\delta_0(\lambda)$ could be $O(1)$ we have to keep all terms.

Domenant balance suggests:

⊕ either $\phi_0'^2(x) = -\dfrac{\lambda^2}{\delta_0^2(\lambda)} q(x)$

$$\Rightarrow \phi_0'(x) = \pm i \frac{|\lambda|}{\delta_0(\lambda)} \sqrt{q(x)}$$

↑ either a complex #
or a real one depending
on sign of $q(x)$

For this to be in balance $\forall \lambda$, we need

$\delta_0(\lambda) = |\lambda|$ then $\phi_0'(x) = \pm i \sqrt{q(x)}$

To see if it works,

$$\rightarrow \underbrace{\left(\pm i \sqrt{q(x)}\right)' |\lambda|}_{O(\lambda)} \underbrace{- \frac{\lambda^2 \cancel{|\lambda|^2}}{\cancel{|\lambda|^2}} q(x) + \lambda^2 q(x)}_{O(\lambda^2)}$$

since $\lambda \ll \lambda^2$ for large λ, the last 2 terms balance ✓

⊕ or $\phi_0'' \delta_0(\lambda) + \lambda^2 q(x) = 0$

$$\Rightarrow \phi_0'' = -\frac{\lambda^2}{\delta_0(\lambda)} q(x) \rightarrow \phi_0' = -\int q(x)$$

and $\delta_0(\lambda) = \lambda^2$ for balance

But then

$$\underbrace{-\frac{\lambda^2}{\delta_0(\lambda)} q(x) \cdot \lambda^2}_{O(\lambda^2)} + \underbrace{\left(\int q(x)dx\right)^2 \cdot \lambda^4}_{O(\lambda^4)} + \underbrace{\lambda^2 q(x)}_{O(\lambda^2)} = 0$$

↳ neglected term is larger than kept terms → bad

{ etc.

We then show that only the solution with

$\phi_0'(x) = \pm i \sqrt{q(x)}$, $\delta_0(\lambda) = |\lambda|$ works.

To the next order,

$$\phi_0'' \delta_0(\lambda) + \phi_1'' \delta_1(\lambda) + \cancel{\phi_0'^2 \delta_0^2(\lambda)} + 2\phi_0'\phi_1' \delta_0(\lambda)\delta_1(\lambda) + \cancel{\lambda^2 q(x)} = 0$$

$$\Rightarrow \quad \phi_1'' + 2\phi_0'\phi_1' \delta_0(\lambda) + \phi_0'' \frac{\delta_0(\lambda)}{\delta_1(\lambda)} = 0.$$

$$\Rightarrow \quad \underbrace{\phi_1''}_{\substack{\text{this is} \\ O(1)}} \pm \underbrace{2i\sqrt{q(x)}\,\phi_1' |\lambda|}_{\substack{\text{this is} \\ O(|\lambda|)}} \pm \left(i\sqrt{q(x)}\right)' \frac{|\lambda|}{\delta_1(\lambda)} = 0.$$

$\ll$

So the dominant balance here must be

$$2i\sqrt{q(x)}\,\phi_1'(x) = -\left(i\sqrt{q(x)}\right)' \cdot \frac{1}{\delta_1(\lambda)}$$

this suggests that $\delta_1(\lambda) = 1$ and that

$$\phi_1'(x) = -\frac{1}{2}\frac{\left(\sqrt{q(x)}\right)'}{\sqrt{q(x)}} \Rightarrow \phi_1(x) = -\frac{1}{2}\ln\sqrt{q(x)} + \text{const.}$$

So finally:

$$y(x) = \exp\left[\pm i|\lambda|\int\sqrt{q(x')}\,dx' - \frac{1}{2}\ln\sqrt{q(x)} + k_\pm\right]$$

$$= \frac{a}{q(x)^{1/4}}\cos\left(|\lambda|\int\sqrt{q(x')}\,dx'\right)$$

$$+ \frac{b}{q(x)^{1/4}}\sin\left(|\lambda|\int\sqrt{q(x')}\,dx'\right)$$

Note that the formula can be applied "as is" if $q(x) > 0 \ \forall x$, must be dealt with great care if $q(x) < 0 \ \forall x$, and fails for $q(x) = 0$ somewhere in interval (see later for more).

<u>Example 1</u> Consider the equation $\frac{d^2y}{dx^2} + \lambda^2(1+x^2)^2 y = 0$

then $q(x) = (1+x^2)^2$

The steps above can be repeated to give

- at the lowest order:

$$\phi_0'' S_0(\lambda) + \phi_0'^2(x) S_0^2(\lambda) + \lambda^2 (1+x^2)^2 = 0$$

$\Rightarrow$ the dominant balance is

$$\begin{cases} \phi_0'^2(x) = -(1+x^2)^2 \\ S_0^2(\lambda) = \lambda^2 \end{cases}$$

$$\Rightarrow \begin{cases} \phi_0'(x) = \pm i(1+x^2) \\ S_0(\lambda) = |\lambda| \end{cases}$$

$$\Rightarrow \begin{cases} \phi_0(x) = \pm i\left(x + \frac{x^3}{3}\right) + K \\ S_0(\lambda) = |\lambda| \end{cases}$$

- To the next order

$$\phi_0'' S_0(\lambda) + \phi_1'' S_1(\lambda) + 2\phi_0'\phi_1' S_0(\lambda) S_1(\lambda) = 0$$

$\rightarrow$ the dominant balance is

$$\phi_0'' \cancel{|\lambda|} + 2\phi_0'\phi_1' \cancel{|\lambda|} S_1(\lambda) = 0$$

$$\Rightarrow \begin{cases} S_1(\lambda) = 1 \\ \phi_1'(x) = -\frac{\phi_0''}{2\phi_0'} \end{cases}$$

$$\rightarrow \phi_1(x) = -\frac{1}{2}\ln|\phi_0'|$$

$$= -\frac{1}{2}\ln|i(1+x^2)|$$

Note that here it's easier to write

$i = e^{\frac{i\pi}{2} + 2ik\pi}$ so

$$\phi_1(x) = -\frac{1}{2}\left\{\ln|i| + \ln(1+x^2)\right\} = -\frac{1}{2}\left(\frac{i\pi}{2} + 2i\pi k + \ln(1+x^2)\right.$$

Since ϕ_1 is still a function of x which is multiplied by $\delta_1(\lambda) = 1$, $\xrightarrow{}$ (hence does not $\to 0$ as $\lambda \to \infty$) we go to next order

$$\phi_1'' \delta_1(\lambda) + \phi_1'^2 \delta_1(\lambda)^2 + 2\phi_0' \phi_2' \delta_0(\lambda) \delta_2(\lambda) + \phi_2'' \delta_2(\lambda) = 0$$

$$\to \underbrace{\phi_1'' + \phi_1'^2 + 2\phi_0' \phi_2' |\lambda| \delta_2(\lambda)}_{} + \phi_2'' \delta_2(\lambda) = 0$$

dominant balance:

has $\begin{cases} \delta_2(\lambda) \sim \frac{1}{|\lambda|} & \to \text{ this} \to 0 \text{ as } \lambda \to \infty. \\ \phi_2' = \frac{-(\phi_1'' + \phi_1'^2)}{2\phi_0'} \end{cases}$

$\downarrow$ we can stop here

So finally

$$y(x) = \exp\left\{ \pm i|\lambda|\left(x + \frac{x^3}{3}\right) + K_\pm - \frac{1}{2}\ln(1+x^2) + o(\ldots\right.$$

$$\cong \frac{a}{\sqrt{1+x^2}} \cos\left[|\lambda|\left(x + \frac{x^3}{3}\right)\right] + \frac{b}{\sqrt{1+x^2}} \sin\left[|\lambda|\left(x + \frac{x^3}{3}\right)\right]$$

<u>Example 2</u> $\quad \frac{d^2 y}{dx^2} - \lambda \frac{y}{x} = 0 \quad \to$ This equation is regular-singular @ $x=0$

Let's first consider the case $x > 0$:

Lowest order: $\phi_0'' \delta_0(\lambda) + \phi_0'^2 \delta_0^2(\lambda) - \frac{\lambda}{x} = 0$

Suppose $\phi_0'' \delta_0(\lambda) = \frac{\lambda}{x} \Rightarrow \delta_0(\lambda) = \lambda$

but then $\phi_0'^2 \delta_0^2(\lambda) = O(\lambda^2) \gg \lambda$

$\to$ doesn't work

$\{$ etc

The dominant balance is $\phi_0'^2 \delta_0^2(\lambda) = \frac{\lambda}{x}$

$\Rightarrow \delta_0^2(\lambda) = \pm\lambda$ and $\phi_0'^2 = \pm\frac{1}{x}$

if $\lambda > 0$: $\delta_0(\lambda) = \sqrt{\lambda}$ and $\phi_0'^2 = \frac{1}{x} \Rightarrow \phi_0'(x) = \frac{\pm 1}{\sqrt{x}}$

$$\Rightarrow \phi_0(x) = \pm 2\sqrt{x} + K$$

if $\lambda < 0$: $\delta_0(\lambda) = \sqrt{-\lambda}$ and $\phi_0'^2 = -\frac{1}{x} \Rightarrow \phi_0'(x) = \frac{\pm i}{\sqrt{x}}$

$$\Rightarrow \phi_0(x) = \pm 2i\sqrt{x} + K$$

In both cases we see that

$$\delta_0(\lambda)\,\phi_0(x) = \pm 2\sqrt{\lambda x} + K$$

(whether $\lambda > 0$ or < 0).

To the next order we have

$$\phi_0''\,\delta_0(\lambda) + \phi_1''\,\delta_1(\lambda) + 2\phi_0'\phi_1'\,\delta_0(\lambda)\delta_1(\lambda) = 0$$

$\Rightarrow$ dominant balance is

$$\phi_0'' + 2\phi_0'\phi_1'\,\delta_1(\lambda) = 0$$

$$\Rightarrow \begin{cases} \phi_1'(x) = -\dfrac{\phi_0''}{2\phi_0'} \\ \delta_1(\lambda) = 1 \end{cases} \Rightarrow \phi_1(x) = -\frac{1}{2}\ln|\phi_0'|$$

$$\begin{cases} = -\frac{1}{2}\ln\left|\frac{1}{\sqrt{x}}\right| & \lambda > \\ = -\frac{1}{2}\ln\left|\frac{i}{\sqrt{x}}\right| & \lambda < \end{cases}$$

$\hookrightarrow$ as before the $\ln|i|$ is just an additive co

To the next order,

$$\phi_1''\,\delta_1(\lambda) + \phi_2''\,\delta_2(\lambda) + \phi_1'^2\,\delta_1^2(\lambda) + 2\phi_0'\phi_2'\,\delta_0(\lambda)\delta_2(\lambda) = 0$$

$$\Rightarrow \underbrace{\phi_1'' + \phi_1'^2 + 2\phi_0'\phi_2'\sqrt{\lambda}\,\delta_2(\lambda)}_{\text{dominant balance so}} + \phi_2''\,\delta_2(\lambda) = 0$$

$$\delta_2(\lambda) = O\left(\frac{1}{\sqrt{\lambda}}\right).$$

$\Rightarrow$ finally, for $x > 0$,

$$y(x) = \exp\left\{ \pm 2\sqrt{\lambda x} - \frac{1}{2}\ln\left|\frac{1}{\sqrt{x}}\right| + K_\pm + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}$$

$\rightarrow$ if $\lambda > 0$:

$$y(x) = x^{1/4} \left\{ a e^{2\sqrt{\lambda x} + O\left(\frac{1}{\sqrt{\lambda}}\right)} + b e^{-2\sqrt{\lambda x} + O\left(\frac{1}{\sqrt{\lambda}}\right)} \right\}$$

if $\lambda < 0$

$$y(x) = x^{1/4} \left\{ a \cos\left(2\sqrt{\lambda x} + O\left(\frac{1}{\sqrt{\lambda}}\right)\right) + b \sin\left(2\sqrt{\lambda x} + O\left(\frac{1}{\sqrt{\lambda}}\right)\right) \right\}$$

The case $x < 0$ is very much the same except that the solutions for $\lambda > 0$ and $\lambda < 0$ are switched.

Note: In the two expamples above, the solution is valid for all x $\rightarrow$ so we can actually apply boundary conditions to it.

Cases where $q(x)$ vanishes somewhere in the interval considered are trickier. Let's look at one in particular:

Example 3 The Airy eigenvalue eq: $\frac{d^2y}{dx^2} + \lambda^2 x y = 0$ for $x \geq 0$

To lowest order we get

$$\phi_0'' \delta_0(\lambda) + \phi_0'^2 \delta_0^2(\lambda) + \lambda^2 x = 0$$

$$\rightarrow \begin{cases} \phi_0'^2 = -x \\ \delta_0^2(\lambda) = |\lambda| \end{cases} \rightarrow \phi_0'(x) = \pm i\sqrt{x} \rightarrow \phi_0(x) = \pm \frac{2}{3} i x^{3/2} + K$$

To the next order:

$$\phi_0'' \delta_0(\lambda) + \phi_1'' \delta_1(\lambda) + 2\phi_0'\phi_1' \delta_0(\lambda)\delta_1(\lambda) = 0$$

$\rightarrow$ problem: $\phi_0'' = \pm \frac{i}{2\sqrt{x}} \rightarrow \infty$ as $x \rightarrow 0$

$\phi_0' \rightarrow 0$ as $x \rightarrow 0$

so at $x=0$ the dominant balance must be between $\underbrace{\phi_0'' S_0(\lambda) + \phi_1'' S_1(\lambda)}$

but these are $\neq$ orders in λ by definition!

$\rightarrow$ problem.

If we just pick the solution away from $x=0$ & hope for the best we get:

$$\begin{cases} \phi_1'(x) = -\frac{1}{2}\frac{\phi_0''}{\phi_0'} & \text{as usual} \rightarrow \phi_1(x) = -\frac{1}{2}\ln|\phi_0'| \\ S_1(\lambda) = 1 & \qquad\qquad = -\frac{1}{2}\ln|i\sqrt{x}| \end{cases}$$

$$\rightarrow \quad y(x) = \exp\left\{ \pm \frac{2}{3} i x^{3/2} + K_\pm - \frac{1}{2}\ln\sqrt{x} \right\}$$

$$= \frac{1}{x^{1/4}}\left\{ a\cos\left(\frac{2}{3}x^{3/2} + o\left(\frac{1}{\lambda}\right)\right) + b\sin\left(\frac{2}{3}x^{3/2} + o\left(\frac{1}{\lambda}\right)\right)\right\}$$

$\rightarrow$ this is singular as $x \to 0$, and our approximate solution may not be valid there

A point where $q(x) = 0$ is called a "turning point". These points must be dealt with more carefully (see later for detail)