CHAPTER 6: The method of dominant balance & the WKB approximation

I Regular, singular, and regular-singular points; Froebenius series

In all that follows (except where explicitly mentioned), we now consider the general ODE

$$\frac{d^2 f}{dx^2} + p(x)\frac{df}{dx} + q(x)f = 0$$

subject to some BCs.

① Definitions

Consider a point x_0 in the interval between the BCs

- if $p(x_0)$ AND $q(x_0)$ are finite (i.e not $\pm\infty$) then x_0 is a regular point, otherwise x_0 is a singular point
- If x_0 is singular, but BOTH
$$\lim_{x \to x_0} (x-x_0)\,p(x) \quad \text{AND} \quad \lim_{x \to x_0} (x-x_0)^2 q(x)$$
are finite (as above) then x_0 is a regular singular point; otherwise x_0 is called an irregular singular point

Example 1: the equation $\frac{d^2 f}{dx^2} = xf$ is called an Airy Equation

$$\begin{cases} p(x) = 0 & \text{for all points} \\ q(x) = x & \end{cases}$$

$\to$ all points are regular except $x \to \pm\infty$

To find out the behavior near ∞, let $s = \frac{1}{x}$ and study the behavior as $s \to 0$

$$\frac{d}{dx} = \frac{ds}{dx}\frac{d}{ds} = -\frac{1}{x^2}\frac{d}{ds} = -s^2\frac{d}{ds}$$

$$\frac{d^2}{dx^2} = -s^2\frac{d}{ds}\left(-s^2\frac{d}{ds}\right) = s^2\frac{d}{ds}\left(s^2\frac{d}{ds}\right)$$

$$= s^4\frac{d^2}{ds^2} + 2s^3\frac{d}{ds}$$

$$\Rightarrow \quad s^4\frac{d^2f}{ds^2} + 2s^3\frac{df}{ds} - \frac{1}{s}f = 0$$

$$\Rightarrow \quad \frac{d^2f}{ds^2} + \frac{2}{s}\frac{df}{ds} - \frac{1}{s^5}f = 0$$

$$p(s) = \frac{2}{s} \quad \text{and} \quad q(s) = -\frac{1}{s^5}$$

$\lim_{s\to 0} p(s)$ and $\lim_{s\to 0} q(s)$ are infinite.

$\lim_{s\to 0} sp(s) = 2$ but $\lim_{s\to 0} q(s)$ is infinite

$\Rightarrow$ The point at ∞ is an irregular-singular pt. for the Airy function

Example 2. Bessel Equation $\quad x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + x^2 f = 0$

$\rightarrow \quad \frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} + f = 0$; all points are regular except $x = 0$. $\left(p(x) = \frac{1}{x},\ q(x) = 1\right)$

$\lim_{x\to 0} xp(x) = 1 \qquad \lim_{x\to 0} x^2 q(x) = 0 \qquad$ both finite

$\Rightarrow$ 0 is a regular-singular point of the Bessel Equation.

Example 3 $\quad \frac{d^2f}{dx^2} + \frac{f}{x^4} = 0$

$x = 0$ is irregular singular. By contrast, Near $x = \infty$, however this becomes

$\frac{d^2f}{ds^2} + \frac{2}{s}\frac{df}{ds} + f = 0 \quad \Rightarrow \quad s = 0$ $(x = \infty)$ is regular-singu

(2) Frobenius series

Theorem:

Suppose $x=0$ is a regular point or a regular-singular point. Then at least one solution of the ODE $\frac{d^2f}{dx^2}+p(x)\frac{df}{dx}+q(x)=0$ has the form

$$f_1(x) = x^{\alpha}\sum_{k=0}^{\infty} a_k x^k \quad \text{where } a_0 \neq 0$$

This solution is called the Frobenius series.

Notes:

- In general, both solutions of the ODE have this form. However, in some circumstances (see below) the second solution instead has the form

$$f_2(x) = b\ln|x|\, f_1(x) + x^{\alpha_2}\sum_{k=0}^{\infty} b_k x^k.$$

where b is constant $\neq 0$.

(Note that with $b=0$ we recover the first form).

- For an irregular-singular point, this does not work and a $\neq$ method must be used (see later)

However, one can also recast the equation in terms of $s=\frac{1}{x}$, and study the ODE in s near $s=0$ $\rightarrow$ get a Frobenius series in s if $s=0$ is a regular or regular-singular pt.

- While the ∞ series usually converges, the truncated series's accuracy depends on the distance to the closest irregular-singular pt.
- We can also construct series near x_0 where $x_0 \neq 0$

Example 1 Spherical Bessel functions of order 0.

The equation for this function is $\frac{d^2f}{dx^2} + \frac{2}{x}f + f = 0$

we saw before that is is regular everywhere except $x=0$; $x=0$ is a regular-singular pt.

We then try $f(x) = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$

$$\Rightarrow \frac{df}{dx} = \sum_{k=0}^{\infty} a_k(\alpha+k)x^{\alpha+k-1}$$

$$\frac{d^2f}{dx^2} = \sum_{k=0} a_k(\alpha+k)(\alpha+k-1)x^{\alpha+k-2}$$

so the equation becomes:

$$\sum_{k=0}^{\infty} a_k(\alpha+k)(\alpha+k-1)x^{\alpha+k-2} + 2\sum_{k=0} a_k(\alpha+k)x^{\alpha+k-2} + \sum_{k=0}^{\infty} a_k x^{\alpha+k} = 0$$

Dividing by x^α, & equating orders of this function of x, we get

to order x^{-2}: $\cancel{a_0}\alpha(\alpha-1) + 2\cancel{a_0}\alpha = 0$

to order x^{-1}: $a_1(\alpha+1)\alpha + 2a_1(\alpha+1) = 0$

to order x^n $(n \geq 0)$: $a_{n+2}(\alpha+n+2)(\alpha+n+1) + 2a_{n+2}(\alpha+n+2) + a_n = 0$

From the lowest order we get $\alpha(\alpha+1) = 0 \Rightarrow$

$\underline{\alpha = 0}$ or $\underline{\alpha = -1}$

→ This yields the 2 fundamental solutions.

Let's study them separately:

$\underline{\alpha = 0}$: a_0 is arbitrary (that's our integration constant)

- The order x^{-1} yields: $2a_1 = 0 \Rightarrow a_1 = 0$
- The next orders yield:

$$a_{n+2}(\alpha+n+2)(\alpha+n+3) = -a_n \qquad \text{(in general)}$$

$$\Rightarrow a_{n+2} = -\frac{a_n}{(n+2)(n+3)}$$

so: $\quad a_2 = -\frac{a_0}{3!} \qquad a_3 = 0 \qquad a_4 = -\frac{a_2}{4\times 5} = \frac{a_0}{5!}$

... etc

So the first fundamental solution is:

$$f_1(x) = a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

$$= \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) = \frac{a_0 \sin x}{x}.$$

$\underline{\alpha = -1}$: The order x^{-1} yields nothing $\Rightarrow a_1$ arbitrary as well

$\rightarrow$ Next orders: $\quad a_{n+2} = \frac{-a_n}{(\alpha+n+2)(\alpha+n+3)} = -\frac{a_n}{(n+1)(n+2)}$

so $\quad a_2 = -\frac{a_0}{2} \qquad a_3 = -\frac{a_1}{3!}$

$$a_4 = -\frac{a_2}{3\times 4} = \frac{a_0}{4!} \qquad a_5 = -\frac{a_3}{4\times 5} = \frac{a_1}{5!}$$

etc...

$$\Rightarrow \quad f_2(x) = \frac{a_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \leftarrow \text{new solution}$$

$$+ \frac{a_1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \leftarrow \text{old solution}$$

$$= \frac{a_0 \cos x}{x} + \frac{a_1 \sin x}{x}$$

$\rightarrow$ we see that the spherical Bessel eq. has 2 fundamental solutions: $j_0(x) = \frac{\sin x}{x}$, $y_0(x) = \frac{\cos x}{x}$

Example 2 A More general case (slightly harder problem)

Consider the equation $\frac{d^2f}{dx^2} + p(x)\frac{df}{dx} + q(x)f = 0$

where both $p(0)$ & $q(0)$ are finite and non-zero.

- $x=0$ is a regular point, so we can try the Frobenius series:

$$f(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}$$

as before:
$$\left\{\begin{aligned} \frac{df}{dx} &= \sum_{k=0}^{\infty} a_k(k+\alpha)x^{k+\alpha-1} \\ \frac{d^2f}{dx^2} &= \sum_{k=0}^{\infty} a_k(k+\alpha)(k+\alpha-1)x^{k+\alpha-2} \end{aligned}\right.$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k(k+\alpha)(k+\alpha-1)x^{k+\alpha-2} + p(x)\sum_{k=0}^{\infty} a_k(k+\alpha)x^{k+\alpha-1} + q(x)\sum_{k=0}^{\infty} a_k x^{k+\alpha} = 0$$

Since $x=0$ is regular, we can Taylor expand p & q without difficulty. This yields:

$$p(x) = p(0) + xp'(0) + \frac{x^2}{2}p''(0)\ldots = \sum_j p^{(j)}(0)\frac{x^j}{j!}$$

and similarly for q.

Plugging this in, and dividing by x^α, we get:

$$\sum_{k=0} a_k(k+\alpha)(k+\alpha-1)x^{k-2} + \sum_{j,k=0}^{\infty}(k+\alpha)p^{(j)}(0)\,a_k\frac{x^{k+j-1}}{j!} + \sum_{j,k=0}^{\infty} q^{(j)}(0)\,a_k\frac{x^{k+j}}{j!} = 0$$

$\rightarrow$ Equating powers of x, we then get:

$O(x^{-2})$: $\quad a_0\alpha(\alpha-1) = 0$

$O(x^{-1})$: $\quad a_1(\alpha+1)\alpha + \alpha p(0)a_0 = 0$

$$O(x^0): \quad a_2(2+\alpha)(1+\alpha) + (1+\alpha)p(0)a_1 + \alpha p'(0)a_0 + q(0)a_0 = 0$$

$n \geq 0$ $$O(x^n): \quad a_{n+2}(n+2+\alpha)(n+1+\alpha) + \sum_{k=0}^{n+1} \frac{(\alpha+k)p^{(n+1-k)}(0)a_k}{(n+1-k)!} + \sum_{k=0}^{n} \frac{q^{(n-k)}(0)a_k}{(n-k)!} = 0$$

From the lowest order we get (since $a_0 \neq 0$ by definition)

$\alpha = 0$ <u>OR</u> $\alpha = 1$ $\Rightarrow$ This gives us our 2 fundamental solutions.

In the case $\alpha = 0$: a_0 is arbitrary (that's the integration constant)

- the $O(x^{-1})$ equation doesn't tell us anything
- the $O(x^0)$ equation states

$$2a_2 + p(0)a_1 + q(0)a_0 = 0$$

$\rightarrow$ choose (arbitrarily) $a_1 = 0 \Rightarrow$

$$a_2 = -\frac{q(0)}{2}a_0$$

- the $O(x^n)$ equation gives us the recurrence to get all solutions:

$$a_{n+2}(n+2)(n+1) = -\sum_{k=0}^{n+1} \frac{k p^{(n+1-k)}(0)a_k}{(n+1-k)!} \sum_{k=0}^{n} \frac{q^{(n-k)}(0)a_k}{(n-k)!}$$

$$\forall n \geq 0$$

We can then do the same for $\alpha = 1$:

- The $O(x^{-1})$ equation states that:

$$2a_1 + p(0)a_0 = 0 \Rightarrow a_1 = -\frac{p(0)a_0}{2}$$

- The $O(x^0)$ equation states that:

$$6a_2 + 2p(0)a_1 + p'(0)a_0 + q(0)a_0 = 0$$

$\Rightarrow$ yields a_2, etc ...

Note: were we allowed to pick $a_1 = 0$ in the first case?

As it turns out, it's exactly the same issue as for the spherical Bessel function. Suppose we had not picked $a_1 = 0$

Then, in the first solution we would set

$$a_2 = -\frac{q(0)}{2} a_0 - \frac{p(0)}{2} a_1$$

$\vdots$

etc.

$\Rightarrow$ The lowest orders of $f_1(x)$ can be written as

$$f_1(x) = a_0 + a_1 x + \left(-\frac{q(0)}{2} a_0 - \frac{p(0)}{2} a_1\right) x^2 + \cdots$$

Meanwhile, the second solution has

$$f_2(x) = x\left[\tilde{a}_0 - \frac{p(0)\tilde{a}_0}{2} x + \cdots\right]$$

we see that $f_2(x)$ is in fact contained in the $f_1(x)$ solution (in the terms in a_1)

$\Rightarrow$ either we set a_1 to 0 & then we have to find both solutions

$\Rightarrow$ or we don't set a_1 to 0 & then "$f_1(x)$" actually contains both fundamental solutions

Example 3: (an even harder problem): The Bessel Equation:

$$\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} + f = 0$$ → has a regular singular pt.

We assume $f(x) = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$

⇒ expand $\frac{df}{dx}$, $\frac{d^2f}{dx^2}$ as usual →

$$\sum_{k=0}^{\infty} a_k (\alpha+k)(\alpha+k-1)x^{\alpha+k-2}$$
$$+ \sum_{k=0}^{\infty} a_k(\alpha+k)x^{\alpha+k-2} + \sum_{k=0}^{\infty} a_k x^{\alpha+k} = 0.$$ → divide by x

2 Equate powers of x, as usual. ⇒

to $O(x^{-2})$: $a_0(\alpha-1)\alpha + a_0\alpha = 0 \Rightarrow a_0\alpha^2 = 0$

to $O(x^{-1})$: $a_1(\alpha+1)\alpha + a_1(\alpha+1) = 0 \Rightarrow a_1(\alpha+1)^2 = 0$

to $O(x^n)$: $a_{n+2}(\alpha+n+2)(\alpha+n+1) + a_{n+2}(\alpha+n+2) + a_n = 0$

The lowest order equation suggests that $\alpha = 0$, but this time it's a double-root (so we will only get one solution from this).

Let's have $\alpha = 0$: → a_0 arbitrary.

- from $O(x^{-1}) \Rightarrow a_1 = 0$
- from $O(x^n) \Rightarrow a_{n+2} = -\frac{a_n}{(\alpha+n+2)^2} = -\frac{a_n}{(n+2)^2}$

$$\text{so } f_1(x) = a_0 - \frac{a_0}{4}x^2 + \frac{a_0 x^4}{64} - \cdots$$
$$= a_0\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right)$$

To get the next solution, we remember that it would take the form

$$f_2(x) = b\ln|x|\, f_1(x) + \sum^{\infty} b_k x^{\alpha_2+k}$$

Note that since a_0 is arbitrary, we can also arbitrarily choose $b_0 = 1$ (and fold it into a_0)

$$\Rightarrow \quad f_2(x) = \ln|x|\, f_1(x) + \sum_{k=0}^{\infty} b_k x^{\alpha_2+k}$$

$$\frac{df_2}{dx} = \frac{1}{x} f_1 + \ln|x| \frac{df_1}{dx} + \sum_{k=0}^{\infty} b_k (\alpha_2+k) x^{\alpha_2+k-1}$$

$$\frac{d^2f_2}{dx^2} = -\frac{1}{x^2} f_1 + \frac{2}{x}\frac{df_1}{dx} + \ln|x| \frac{d^2f_1}{dx^2}$$

$$+ \sum_{k=0}^{\infty} b_k (\alpha_2+k)(\alpha_2+k-1) x^{\alpha_2+k-2}$$

Plugging this in, we then get (collecting similar terms):

$$\ln|x| \underbrace{\left[\cancel{\frac{d^2 f_1}{dx^2} + \frac{1}{x}\frac{df_1}{dx} + f_1} \right]}_{=0} + \frac{2}{x}\frac{df_1}{dx} - \cancel{\frac{1}{x^2} f_1} + \cancel{\frac{1}{x^2} f_1}$$

$$+ \sum_{k=0}^{\infty} b_k (\alpha_2+k)(\alpha_2+k-1) x^{\alpha_2+k-2} + \sum_{k=0}^{\infty} b_k(\alpha_2+k) x^{\alpha_2+k-2} + \sum_{k=0}^{\infty} b_k x^{\alpha_2+k} =$$

The second line is exactly what we had earlier for the α, and a_k variables. It would give the same solution were it not for the leftover term on the first line, which has $\frac{2}{x}\frac{df_1}{dx} = 2\sum_{k=1}^{\infty} a_k\, k\, x^{k-2}$

$\Rightarrow$ Now, dividing by α_2, we are left with:

$$\sum_{k=0}^{\infty} b_k (\alpha_2+k)(\alpha_2+k-1) x^{k-2} + \sum_{k=0}^{\infty} b_k (\alpha_2+k) x^{k-2}$$

$$+ \sum_{k=0}^{\infty} b_k x^k + 2\sum_{k=1}^{\infty} a_k\, k \quad x^{k-2-\alpha_2} = 0$$

- Expanding term by term & keeping only the lowest few in each case we get:

$$b_0 \alpha_2(\alpha_2-1) x^{-2} + b_1(\alpha_2+1)\alpha_2 x^{-1} + \dots + b_0 \alpha_2 x^{-2} + b_1(\alpha_2+1) x^{-1} + \dots$$

$$+ \quad b_0 + b_1 x + \dots \quad + \underbrace{\cancel{2a_1 x^{-1-\alpha_2}}}_{\text{since } a_1 = 0} + 2a_2 \cdot 2x^{-\alpha_2} + \dots = 0$$

We see that there are a few possibilities.

- $-\alpha_2 < -2$ in which case we would need $a_2 = 0$
  $(\alpha_2 > 2)$ $\rightarrow$ inconsistent since $a_0 \neq$
- $-\alpha_2 = -2$ $(\alpha_2 = 2)$ in which case we need

(to $O(x^{-2})$): $b_0 \alpha_2(\alpha_2 - 1) + b_0 \alpha_2 + 4a_2 = 0$

$\rightarrow b_0 \alpha_2^2 + 4a_2 = 0$ $\rightarrow$ this gives $b_0 = -a$

(to $O(x^{-1})$): $b_1(\alpha_2+1)\alpha_2 + b_1(\alpha_2+1) + 6a_3 = 0$

$b_1(\alpha_2+1)^2 + 6a_3 = 0 \rightarrow$ this gives $b_1 = -\frac{6a_3}{9} = 0$

(etc)

this works & gives a unique solution.

- $-\alpha_2 > -2$ in which case the lowest order is
  $(\alpha_2 < 2)$ $b_0 \alpha_2(\alpha_2 - 1) + b_0 \alpha_2 = 0 \Rightarrow b_0 \alpha_2^2 = 0$

  so $\underline{\alpha_2 = 0}$ b_0 arbitrary

$\rightarrow$ to $O(x^{-1})$: $\cancel{b_1(\alpha_2+1)\alpha_2} + b_1(\alpha_2+1) + 2a_1 = 0$

$\rightarrow b_1 = -2a_1 = 0$

to $O(x^0)$: $b_2(\alpha_2+2)(\alpha_2+1) + b_2(\alpha_2+2) + b_0 + 4a_2 = 0$

$\Rightarrow 4b_2 + b_0 = -4a_2$

$\Rightarrow b_2 = -\frac{4a_2 - b_0}{4}$

$= -a_2 - \frac{b_0}{4}$

this works & gives a unique solution.

Case $\alpha = 2$: $f_2(x) = \ln|x| f_1(x) + x^2(-a_2 - 0(x^2) + \dots)$

this solution same as this

Case $\alpha = 0$: $f_2(x) = \ln|x| f_1(x) + b_0 - \left(a_2 + \frac{b_0}{4}\right)x^2 + \dots$

these solutions actually recover f

$\rightarrow$ can set b_0 to 0 if we keep f_1

So finally, we get that the general solution will be a linear combination of $f_1(x)$ & $f_2(x)$ where

$$f_1(x) = a_0\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right) \leftarrow \text{the regular solution } J(x)$$

$$f_2(x) = b_0\left(\ln|x|\, f_1(x) - a_2 x^2 + \cdots\right) \leftarrow \text{the singular solution } Y(x).$$

<u>Example 4:</u> Frobenius series for an irregular-singular pt?

We now study what happens when we try to expand around an irregular singular pt.

Consider $\dfrac{d^2f}{dx^2} + \dfrac{f}{x^4} = 0$ near $x=0$

$\rightarrow$ if we try $f(x) = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$ then

$$\rightarrow \sum_{k=0}^{\infty} a_k (\alpha+k)(\alpha+k-1) x^{\alpha+k-2} + \sum_{k=0}^{\infty} a_k x^{\alpha+k-4} = 0$$

$\rightarrow$ the lowest term is $O(x^{-4})$ but it is not compensated by any term anywhere else $\Rightarrow$ this would require $a_0 = 0$, but by definition we must have $a_0 \neq 0$ $\rightarrow$ inconsistency

This shows that the Frobenius series don't work around an irregular-singular pt.

However, if we remember that with $s = \frac{1}{x}$, the equation becomes $\dfrac{d^2f}{ds^2} + \dfrac{2}{s}\dfrac{df}{ds} + f = 0$, which happens to have a Frobenius series, & eventually the solutions $f(s) = \dfrac{a_0}{s}\cos s + \dfrac{b_0}{s}\sin s$

$$\rightarrow f(x) = \underbrace{a_0 x \cos\left(\tfrac{1}{x}\right) + b_0 x \sin\left(\tfrac{1}{x}\right)}$$

It should be obvious from this that this function is not differentiable @ $x=0$ $\Rightarrow$ cannot have a series near there!