CHAPTER I : Introduction

Most equations that arise from actual problems in science or engineering are too complicated to solve analytically. This is true whether the equation is algebraic, differential, or contains complicated integrals. In most cases, the only way to solve the equation is by using numerical methods.

However, if the equation contains a small parameter that multiplies, or is embedded in the term that makes the equation particularly complex, then it may be possible to approximate the solution by a series of terms, each of which can be obtained analytically.

This course is about how to do this in practice.

This first chapter will give you a taste of the various methods used and of the problems that may arise. These problems will be discussed in future Chapters.

I.1. A model problem: falling through a fluid (air, water, etc.)

a Posing the problem

Consider an object of mass m, dropped from height h_0, from rest, and falling through a fluid whose drag coefficient is λ.

The equation of motion: $\underset{\text{forces}\uparrow}{\underline{F}} = m\underset{\uparrow\text{acceleration}}{\underline{a}}$ becomes, when projected onto direction of gravity $\underline{g}$, is

$$ma = F = -mg - \lambda \underset{\uparrow\text{ vertical velocity}}{w}$$

or in other words

$$m\frac{dw}{dt} = -mg - \lambda w.$$

$$\rightarrow \quad \boxed{\frac{dw}{dt} = -g - \frac{\lambda}{m}w.} \quad (1)$$

This problem can be solved analytically, but suppose for the moment that we forgot about the method of separation of variables, and only know how to solve ODEs of the kind $\frac{dw}{dt} = f(t)$ where f is known. How can we solve (1) analytically in that case?

Earlier, we learned that it may be possible to do so if the equation contains a "small parameter". But what does small mean? Small compared with what? 1 cm is small compared with 1 km but 1 cm is huge compared with 1 μm ....

How do we know something is small?

b Non-dimensionalization & dimensional analysis

The problem with comparing dimensional quantities is that they can be large when expressed in one set of units (e.g. $1\ cm = 10^{-5}\ km = 10^{4}\ \mu m$) but small in another. The only way to determine if something is large or small is to first create a set of units appropriate to the problem at hand and only then look at what is large or small in that set of units. This is the idea behind dimensional analysis & non-dimensionalization.

Let's explore the typical units in our system. Our dimensional input parameters are

m = the mass of the object (in unit of mass)

g = the local gravity (in unit of distance per unit time squared)

h_0 = the initial height (in unit distance)

λ = the drag coefficient (in units of mass per unit time)

Since there is only one characteristic mass-scale in the problem, we should adopt this as our unit mass → from here on, m is the unit mass.

Similarly, h_0 is a natural choice for a unit height/distance → from now on, h_0 is the unit distance.

The problem, however, does not have a natural unit of time. One can be created, however, by combining one of the remaining parameters with the existing ones already chosen: for instance,

- the combination gh_0^{-1} has units of ~~distance squared~~ over time squared
  $\rightarrow$ a natural unit of time could be $\sqrt{\frac{h_0}{g}}$
- the combination $\frac{\lambda}{m}$ has units of one over time $\rightarrow$ $\frac{m}{\lambda}$ could be another unit time.

In fact, both of these units of time are meaningful: one is the dynamical timescale the other one is the frictional damping timescale (more on this later).

Let us now non-dimensionalize all quantities, first based upon the dynamical timescale, and then based upon the frictional damping timescale.

In the first case, we let

unit time $\rightarrow [t] = \sqrt{\frac{h_0}{g}}$ so we define $\hat{t} = \frac{t}{[t]}$ = dimensionless time

unit distance $\rightarrow [d] = h_0$ " " " $\hat{z} = \frac{z}{[d]}$ = dimensionless coordinate z

unit velocity $\rightarrow [v] = \frac{[d]}{[t]} = \frac{h_0}{\sqrt{\frac{h_0}{g}}} = \sqrt{gh_0}$ " " " " $\hat{w} = \frac{w}{[v]}$ = dimensionless velocity

The parameters do not actually need to be re-scaled; as we shall see, this comes out naturally.

→ From $\frac{dw}{dt} = -g - \frac{\lambda}{m} w$

we get
$$\frac{\sqrt{gh_0}\, d\hat{w}}{\sqrt{\frac{h_0}{g}}\, d\hat{t}} = -g - \frac{\lambda}{m} \cdot \sqrt{gh_0}\, \hat{w}$$

$$\rightarrow \quad g \frac{d\hat{w}}{d\hat{t}} = -g - \frac{\lambda}{m} \sqrt{gh_0}\, \hat{w}$$

$$\rightarrow \quad \frac{d\hat{w}}{d\hat{t}} = -1 - \frac{\lambda}{m} \sqrt{\frac{h_0}{g}}\, \hat{w}$$

The equation now depends on a <u>single</u> combination of parameters, $\frac{\lambda}{m} \cdot \sqrt{\frac{h_0}{g}}$. It's easy to verify that this combination is dimensionless. In fact, it is equal to the ratio of the dynamical time to the frictional damping time.

Let's now try it the other way:

unit time $= [t] = \frac{m}{\lambda} \rightarrow \hat{t} = \frac{t}{[t]}$

unit distance $= [d] = h_0 \rightarrow \hat{z} = \frac{z}{h_0}$

unit velocity $= [v] = \frac{[d]}{[t]} \rightarrow \hat{w} = \left(\frac{h_0 \lambda}{m}\right)^{-1} w$

So
$$\frac{dw}{dt} = -g - \frac{\lambda}{m} w$$

$$\rightarrow \quad \frac{\left(\frac{m}{h_0 \lambda}\right)^{-1}}{\frac{m}{\lambda}} \frac{d\hat{w}}{d\hat{t}} = -g - \frac{\lambda}{m} \cdot \frac{h_0 \lambda}{m} \hat{w}$$

$$\rightarrow \quad \frac{h_0 \lambda^2}{m^2} \frac{d\hat{w}}{d\hat{t}} = -g - \frac{h_0 \lambda^2}{m^2} \hat{w}$$

$$\rightarrow \quad \frac{d\hat{w}}{d\hat{t}} = -\frac{gm^2}{h_0 \lambda^2} - \hat{w}$$

Again, the same combination of parameters appear, although this time as a square (and inverse of)

Clearly, this ratio is an important quantity. Let's call it $r = \frac{\lambda}{m}\sqrt{\frac{h_0}{g}}$

→ in the first non-dimensionalization, we have

$$\frac{d\hat{w}}{d\hat{t}} = -1 - r\hat{w}$$

in the second we have

$$\frac{d\hat{w}}{d\hat{t}} = -\frac{1}{r^2} - \hat{w}$$

We therefore see that something interesting could happen

- if r is very small, in which case the first non-dimensionalization tells us that
$$\frac{d\hat{w}}{d\hat{t}} \simeq -1 \qquad \text{(which is easily solved)}$$
- if r is very large, in which case the second non-dimensionalization tells us that
$$\frac{d\hat{w}}{d\hat{t}} \simeq -\hat{w} \qquad \text{(which is also easily solved)}$$

→ depending on whether r or $\frac{1}{r}$ is small we can now find approximate solutions to the original problem.

→ Non-dimensionalizing the equations helps us figure out
- what the <u>relevant</u> combination of parameters are
- whether it is large or small (compared to <u>one</u>)

c. Solving the problem by iterations

In what follows, we assume that r is small, and call it ϵ to remember that it is. From now on, we focus on solving

$$\begin{cases} \dfrac{dw}{dt} = -1 - \epsilon w & \text{(dropping the hats for convenience)} \\ w = 0 \text{ at } t = 0 & \text{(start from rest)} \end{cases}$$

There is a simple way of solving this problem by 'iterations' if ϵ is small.

- In the first iteration, <u>assume</u> $\epsilon = 0$ and solve the problem:

$$\frac{dw^{(0)}}{dt} = -1 \Rightarrow w^{(0)} = -t + K^{(0)}$$

↑ integration constant.

At $t=0$ $w=0$ $\rightarrow$ $w^{(0)} = -t$.

- In the next iteration, use $w^{(0)}$ on the RHS instead of the full w; and solve

$$\frac{dw^{(1)}}{dt} = -1 - \epsilon w^{(0)} = -1 + \epsilon t$$

$$\Rightarrow w^{(1)} = -t + \epsilon \frac{t^2}{2} + K^{(1)}$$

↑ integration constant

notice that $w^{(0)}$ is here

↑ so this is a first-order correction in ϵ

$$w^{(1)} = 0 \text{ at } t=0 \rightarrow K^{(1)} = 0$$

- In the subsequent iterations, solve

$$\frac{dw^{(n)}}{dt} = -1 - \epsilon w^{(n-1)}$$ iteratively

This yields:

$$\frac{dw^{(2)}}{dt} = -1 - \epsilon\left(-t + \epsilon\frac{t^2}{2}\right)$$

$$= -1 + \epsilon t - \frac{\epsilon^2 t^2}{2}$$

$$\rightarrow w^{(2)} = -t + \frac{\epsilon t^2}{2} - \frac{\epsilon^2 t^3}{6} \quad \ldots \quad \text{etc}$$

We see that $w(t)$ can be written as

$$w(t) = s_0(t) + \epsilon s_1(t) + \epsilon^2 s_2(t) + \cdots$$

where s_0, s_1, s_2 are $O(1)$ functions, each multiplied by increasing powers of the small parameter ϵ.

This is called an <u>asymptotic sequence</u> (more on this later). In "good" circumstances, one can hope that this sequence converges to the true solution of the exact problem.

d. <u>Exact solution</u>

In this particular case, in fact, we can actually check what the true solution is as long as we now remember about separation of variables.

$$\frac{dw}{dt} = -1 - \epsilon w \Rightarrow \frac{dw}{1+\epsilon w} = -dt$$

$$\Rightarrow \frac{1}{\epsilon}\ln(1+\epsilon w) = -t + K$$

$$\Rightarrow 1 + \epsilon w = Ce^{-\epsilon t}$$

$$\Rightarrow w = \frac{1}{\epsilon}\left(Ce^{-\epsilon t} - 1\right)$$

To have $w = 0$ at $t = 0$, we need $C = 1$.

So $\quad w(t) = \frac{1}{\epsilon}\left(e^{-\epsilon t} - 1\right)$

As long as $\epsilon t \ll 1$ we can expand the exponential as

$$w(t) \simeq \frac{1}{\epsilon}\left(1 - \epsilon t + \frac{\epsilon^2 t^2}{2} - \frac{\epsilon^3 t^3}{6} \cdots - 1\right)$$

$$\simeq -t + \epsilon\frac{t^2}{2} - \frac{\epsilon^2 t^3}{6} \cdots \rightarrow \text{as before!}$$

(Compare solution w. expansion using gnuplot).
This reveals two things:

- The iterative method works BUT
- it is only valid for times t such that $t\epsilon \ll 1$

Side Note: In fact this is already apparent in the form of $w^{(2)}$ in the previous section: we had

$$w^{(2)} = -t + \frac{\epsilon t^2}{2} - \frac{\epsilon^2 t^3}{6} \cdots$$

$$= -t\underbrace{\left[1 - \frac{\epsilon t}{2} + \frac{\epsilon^2 t^2}{6} + \cdots\right]}_{\text{this only converges if } \epsilon t \ll 1}$$

In the limit where $t \to +\infty$, we have

$$w(t) \to -\frac{1}{\epsilon}$$

This behavior is clearly not captured by the asymptotic expansion...

Here we see one of the standard complications that often occurs in perturbation expansions: that they are only valid for some range of independent variable. Later, we will learn how to fix that problem.

e. Assumed asymptotic sequence

To finish this section, note that there is an alternative way of solving the problem which begins by assuming a solution of the form

$$w(t) = S_0(t) + \epsilon S_1(t) + \epsilon^2 S_2(t) + \cdots$$

and then solving the problem order-by-order.

$$\frac{dw}{dt} = -1 - \epsilon w \qquad w(0) = 0$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\left(S_0 + \epsilon S_1 + \epsilon^2 S_2 \cdots\right) = -1 - \epsilon\left(S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots\right) \\ S_0(0) + \epsilon S_1(0) + \epsilon^2 S_2(0) \cdots = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dS_0}{dt} = -1 & S_0(0) = 0 & \text{to } 0^{th} \text{ order in } \epsilon \\ \frac{dS_1}{dt} = -\epsilon S_0 & S_1(0) = 0 & \text{to } 1^{st} \text{ order in } \epsilon \\ \frac{dS_2}{dt} = -\epsilon S_1 & S_2(0) = 0 & \text{to } 2^{nd} \text{ order in } \epsilon \\ \vdots \end{cases}$$

$$\rightarrow \begin{cases} S_0 = -t \\ S_1 = +\epsilon \frac{t^2}{2} \\ S_2 = -\frac{\epsilon t^3}{6} \end{cases} \rightarrow w(t) = -t + \frac{\epsilon t^2}{2} - \frac{\epsilon t^3}{6} + \cdots$$

as expected.

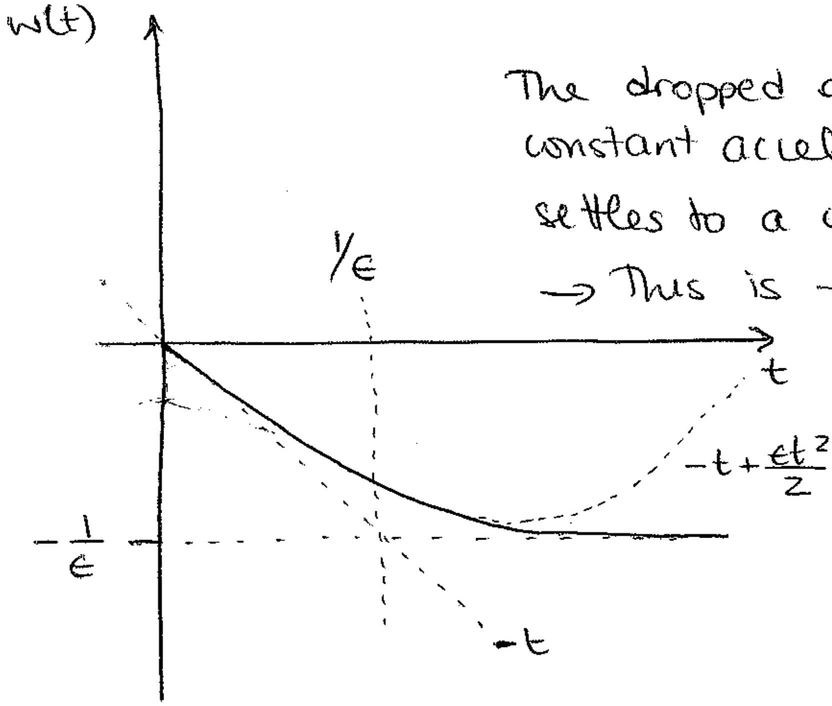
This solution is probably more elegant & easier to justify mathematically than the iterative method. However, it relies on knowing ahead of time what powers of ϵ are contained in the asymptotic sequence. As we shall see later, this is not always obvious.

Physical interpretation of solution

To summarize we found that the true solution is

$$w(t) = \frac{1}{\epsilon}\left(e^{-\epsilon t} - 1\right)$$

and that for small t, $w(t) \simeq -t + \frac{\epsilon^2 t^2}{2}$



The dropped object first has a constant acceleration downward, then settles to a constant velocity $-\frac{1}{\epsilon}$
→ This is the terminal velocity

This happens on the timescale $\frac{1}{\epsilon}$

Going back to dimensional quantities:

$$\text{Terminal velocity} = -\frac{1}{\epsilon}\cdot[V] = -\frac{\sqrt{g h_0}}{\frac{\lambda}{m}\sqrt{\frac{h_0}{g}}} = -\frac{mg}{\lambda}$$

$$\text{Time to relaxation: } \frac{1}{\epsilon}\circ[t] = \sqrt{\frac{h_0}{g}}\cdot\frac{1}{\frac{\lambda}{m}\sqrt{\frac{h_0}{g}}} = \frac{m}{\lambda}$$

= frictional damping timescale.