HW10

Problem 1 p 261 (even)

$$I = \int_0^3 e^{-pt}\, t \sin(t^2)\, dt \simeq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{p^{n+1}}$$

where $f(t) = t\sin(t^2) = t\left(t^2 - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} \cdots\right)$

$$= t^3 - \frac{t^7}{6} + \cdots$$

$$= \frac{f^{(3)}(0)t^3}{3!} + \frac{f^{(7)}(0)}{7!} t^7 \cdots$$

So $I = \frac{f^{(3)}(0)}{p^4} + \frac{f^{(7)}(0)}{p^7} + \text{h.o.t}$

$$= \frac{3!}{p^4} - \frac{7!}{6p^7}$$

$$I = \int_0^\infty \frac{e^{-pt}}{1+t}\, dt \simeq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{p^{n+1}}$$

with $f(t) = \frac{1}{1+t} \simeq 1 - t + t^2 - t^3 + \cdots$

$$\simeq f^{(0)}(0) + f^{(1)}(0)\,t + f^{(2)}(0)\frac{t^2}{2} \cdots$$

$$\Rightarrow I = \frac{f^{(0)}(0)}{p} + \frac{f^{(1)}(0)}{p^2}$$

$$= \frac{1}{p} - \frac{1}{p^2} + \cdots$$

$$\int_0^\pi e^{-pt} \frac{\cos t}{\sqrt{t}}\, dt \simeq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, \Gamma(1-\frac{1}{2}+n)}{n!\; p^{n-\frac{1}{2}+1}}$$ (Use Watson's Lemma this time)

with $a=-\frac{1}{2}$ $\quad f(t) = \cos t \cong 1 - \frac{t^2}{2} + \frac{t^4}{4!} \cdots$

$$\simeq f^{(0)}(0) + \frac{f^{(2)}(0)}{2!} + \frac{f^{(4)}(0)}{4!}$$

So

$$I \simeq \frac{f^{(0)}(0)}{0!} \frac{\Gamma(\frac{1}{2})}{p^{1/2}} + \frac{f^{(2)}(0)}{2!} \frac{\Gamma(5/2)}{p^{5/2}}$$

$$\simeq \frac{\Gamma(\frac{1}{2})}{\sqrt{p}} - \frac{1}{2} \frac{\Gamma(\frac{5}{2})}{p^{5/2}}$$

$$\int_0^1 e^{-pt^2}\, dt$$ let $u = t^2$

$2t\, dt = du \Rightarrow dt = \frac{du}{2\sqrt{u}}$

$$= \int_0^1 e^{-pu} \frac{du}{2\sqrt{u}}$$ $\rightarrow$ Use Watson's Lemma with $a = -\frac{1}{2}$, $f(t) = \frac{1}{2}$

$$= \frac{1}{2} \cdot \frac{1}{0!} \frac{\Gamma(1+0-\frac{1}{2})}{p^{1+0-\frac{1}{2}}} + \text{small terms}$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{2})}{p^{1/2}} = \frac{\sqrt{\pi}}{2\sqrt{p}}$$

Note: this last integral is also exactly

$$\int_0^1 e^{-pt^2} dt = \frac{\sqrt{\pi}}{2\sqrt{p}} \operatorname{erf}(\sqrt{p}) \simeq \frac{\sqrt{\pi}}{2\sqrt{p}}$$ in the limit $p \to +\infty$