HOMEWORK 8

Problem 1: $\frac{d^2y}{dx^2} + \lambda^2(1+x)^2 y = 0 \qquad x \in [0,1]$

$$y(0) = 0, \quad \frac{dy}{dx}(1) = 0.$$

$\rightarrow$ WKB solution has (since $q(x) > 0$)

$$y(x) = \frac{1}{\sqrt{1+x}}\left\{A\cos\left(\lambda\int(1+x')dx'\right) + B\sin\left(\lambda\int(1+x')dx'\right)\right\}$$

$$= \frac{1}{\sqrt{1+x}}\left\{A\cos\left(\lambda\left(x+\frac{x^2}{2}\right)\right) + B\sin\left(\lambda\left(x+\frac{x^2}{2}\right)\right)\right\}$$

$$y(0) = 0 \implies A = 0$$

$$\frac{dy}{dx}(1) = 0 \implies$$

$$\frac{dy}{dx} = -\frac{1}{2}(1+x)^{-3/2}\left\{\cancel{A\cos\left(\lambda\left(x+\frac{x^2}{2}\right)\right)} + B\sin\left(\lambda\left(x+\frac{x^2}{2}\right)\right)\right\}$$

$$+ \frac{1}{\sqrt{1+x}}\left\{\cancel{-\lambda(1+x)A\sin\left(\lambda\left(x+\frac{x^2}{2}\right)\right)}\right.$$

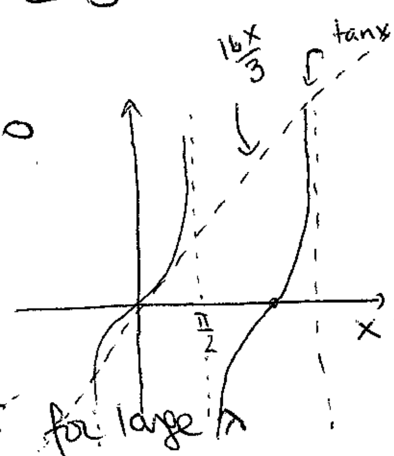
$$\left. + \lambda(1+x)\, B\cos\left(\lambda\left(x+\frac{x^2}{2}\right)\right)\right\}$$

$$\rightarrow \frac{dy}{dx}(1) = -\frac{1}{2}\cdot 2^{-3/2}\left\{B\sin\left(\frac{3}{2}\lambda\right)\right\}$$

$$+ \frac{1}{\sqrt{2}}\left\{2B\lambda\cos\left(\frac{3}{2}\lambda\right)\right\} = 0$$

$$\implies 2\lambda\cos\left(\frac{3}{2}\lambda\right) - \frac{1}{4}\sin\left(\frac{3}{2}\lambda\right) = 0$$

$$\implies \tan\left(\frac{3}{2}\lambda\right) = 8\lambda$$



For large λ, solutions are @ $x = \frac{\pi}{2} + n\pi$

but $x = \frac{3}{2}\lambda \implies \lambda = \frac{2}{3}\left(\frac{\pi}{2} + n\pi\right) = \frac{\pi}{3} + \frac{2n\pi}{3}$ for large λ

- same equation, with $\frac{dy}{dx}(0) = 0$, $\frac{dy}{dx}(1) = 0$

$\rightarrow$ $$\frac{dy}{dx}(0) = -\frac{1}{2}A + \lambda B = 0$$

$\rightarrow$ $$\frac{dy}{dx}(1) = -\frac{1}{2}2^{-3/2}\left(A\cos\left(\lambda\frac{3}{2}\right) + B\sin\left(\frac{3}{2}\lambda\right)\right)$$
$$+\frac{1}{\sqrt{2}}\left\{-2\lambda A\sin\left(\frac{3}{2}\lambda\right) + 2\lambda B\cos\left(\frac{3}{2}\lambda\right)\right\} = 0$$

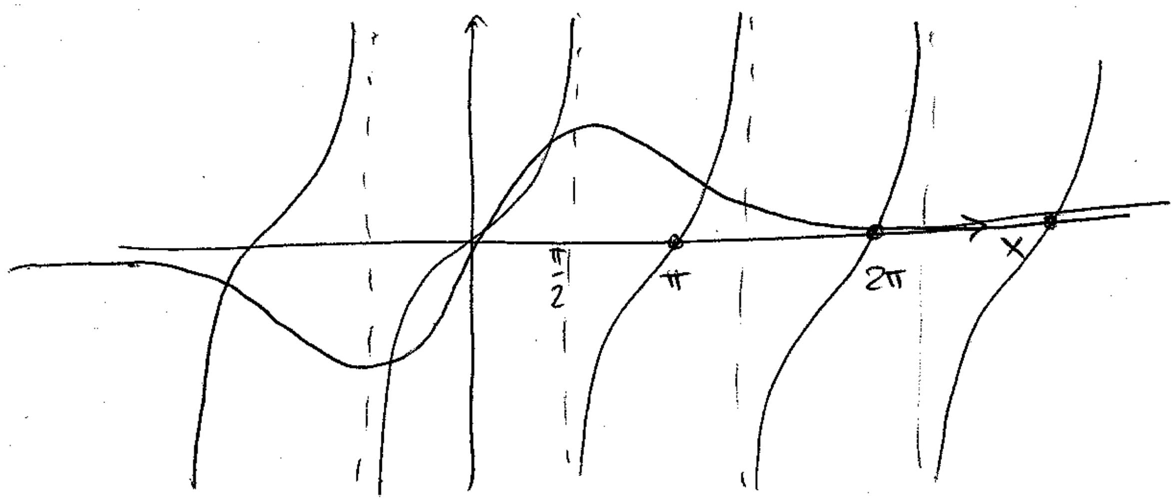
so $A = 2\lambda B$ and

$$-\frac{1}{4}\left(2\lambda \cancel{B}\cos\left(\frac{3}{2}\lambda\right) + \cancel{B}\sin\left(\frac{3}{2}\lambda\right)\right)$$
$$+\left(-2\lambda(2\lambda\cancel{B})\sin\left(\frac{3}{2}\lambda\right) + 2\lambda\cancel{B}\cos\left(\frac{3}{2}\lambda\right)\right) = 0.$$

$$\Rightarrow \cos\left(\frac{3}{2}\lambda\right)\left[-\frac{\lambda}{2} + 2\lambda\right] + \sin\left(\frac{3}{2}\lambda\right)\left[-\frac{1}{4} - 4\lambda^2\right] = 0$$

$$\Rightarrow \tan\left(\frac{3}{2}\lambda\right) = \frac{\frac{3\lambda}{2}}{\frac{1}{4} + 4\lambda^2} \qquad x = \frac{3}{2}\lambda$$

$$\Rightarrow \tan x = \frac{x}{\frac{1}{4} + 4\left(\frac{2}{3}x\right)^2} = \frac{x}{\frac{1}{4} + \frac{16}{9}x^2}$$



$\rightarrow$ solutions tend to $x = n\pi \Rightarrow$
$\frac{3}{2}\lambda = n\pi \Rightarrow \boxed{\lambda = \frac{2n\pi}{3}}$ for large λ

(iv) $$\frac{d^2y}{dx^2} + \lambda^2 e^{4x} y = 0 \qquad x \in (0,1)$$

$$y(0) = y(1) = 0$$

$q(x) > 0 \Rightarrow$ WKB solution is

$$y(x) = \frac{1}{e^x}\left[A\cos\left\{\int \lambda e^{2x'}dx'\right\} + B\sin\left\{\int \lambda e^{2x'}dx'\right\}\right]$$

$$= \frac{1}{e^x}\left[A\cos\left(\frac{\lambda}{2}e^{2x}\right) + B\sin\left(\frac{\lambda}{2}e^{2x}\right)\right]$$

$y(0) = 0 \Rightarrow \quad A\cos\frac{\lambda}{2} + B\sin\frac{\lambda}{2} = 0$

$y(1) = 0 \Rightarrow \quad A\cos\left(\frac{\lambda}{2}e^2\right) + B\sin\left(\frac{\lambda}{2}e^2\right) = 0$

so $\quad A = -B\tan\left(\frac{\lambda}{2}\right)$

so $\quad -\cancel{B}\tan\left(\frac{\lambda}{2}\right)\cos\left(\frac{\lambda}{2}e^2\right) + \cancel{B}\sin\left(\frac{\lambda}{2}e^2\right) = 0$

$\Rightarrow \quad \tan\left(\frac{\lambda}{2}\right) = \tan\left(\frac{\lambda}{2}e^2\right)$

$\Rightarrow \quad \tan x = \tan(e^2 x) \quad$ for $x = \frac{\lambda}{2}$

Since $\tan x$ has period π, this is true for all x such that

$$e^2 x = n\pi + x$$

$$\Rightarrow (e^2 - 1)x = n\pi \Rightarrow x = \frac{n\pi}{e^2-1}$$

$$\Rightarrow \lambda_n = \frac{2n\pi}{e^2-1}$$

Problem 2: $\frac{d^2y}{dx^2} + \lambda^2 x(1+x)^4 y = 0$

This is in Liouville form, so we use the standard WKB solution away from turning pts (here at $x=0$ and $x=-1$)

$$\Rightarrow \quad y(x) = \frac{1}{|x|^{1/4}|1+x|}\left\{ A\cos\int \lambda|x'|^{1/2}(1+x')^2\,dx' + B\sin\int \lambda|x'|^{1/2}(1+x')^2\,dx' \right\}$$ for cases $x(1+x)^4 > 0$

and $$y(x) = \frac{1}{|x|^{1/4}|1+x|}\left\{ A\exp\int \lambda|x'|^{1/2}(1+x')^2\,dx' + B\exp-\int \lambda|x'|^{1/2}(1+x')^2\,dx' \right\}$$ for cases $x(1+x)^4 < 0$

Note: $\int \lambda|x'|^{1/2}(1+x')^2 dx' = \int \lambda y^{1/2}(1-y)^2 dy$ for $y=-x'$ in cases $x'<0$.

① Case $y(-\frac{1}{2}) = 0$, $y(\frac{1}{2}) = 0$

Here the only turning pt in interval is @ $x=0$;

$$\Rightarrow \quad \underset{\substack{\text{(x not close to}\\ 0)}}{y^{\text{outer}}(x)} = \begin{cases} x>0: & \frac{1}{x^{1/4}(1+x)}\left\{ A\cos\left(\lambda\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right)\right) + B\sin\left(\lambda\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right)\right) \right. \\ x<0: & \frac{1}{|x|^{1/4}(1+x)}\left\{ a\exp\left(\lambda\left(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{4}{5}|x|^{\frac{5}{2}} + \frac{2}{7}|x|^{\frac{7}{2}}\right)\right) + b\exp\left(-\lambda\left(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{4}{5}|x|^{\frac{5}{2}} + \frac{2}{7}|x|^{\frac{7}{2}}\right)\right)\right\} \end{cases}$$

Inner solution: let $s = x\lambda^p$

$$\lambda^{2p}\frac{d^2y}{ds^2} + \lambda^{2-p}s(1+s\lambda^{-p})^4 y = 0$$

$\rightarrow$ To lowest order: $\lambda^{2p}\frac{d^2y}{ds^2} + \lambda^{2-p}sy = 0$

VanDyke Least Degeneracy Principle $\Rightarrow 2p = 2-p$

$\Rightarrow p = \frac{2}{3}$

and so $\dfrac{d^2y}{ds^2} + sy = 0$

$\rightarrow \quad y(s) = \alpha \mathrm{Ai}(-s) + \beta \mathrm{Bi}(-s)$

<u>Boundary conditions</u>: • $y(-\tfrac{1}{2}) = 0$

$\Rightarrow \quad a \exp\left(\lambda \cdot \left(\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{3/2} + \tfrac{4}{5}\left(\tfrac{1}{2}\right)^{\frac{5}{2}} + \tfrac{2}{7}\left(\tfrac{1}{2}\right)^{\frac{7}{2}}\right)\right)$

$+ \; b \exp\left(-\lambda\,(\text{same})\right) = 0.$

$\Rightarrow \quad b = a \exp\left(2\lambda\left(\tfrac{2}{3}\tfrac{1}{2\sqrt{2}} - \tfrac{4}{5}\cdot\tfrac{1}{4\sqrt{2}} + \tfrac{2}{7}\cdot\tfrac{1}{8\sqrt{2}}\right)\right)$

$b = a\exp\left(\tfrac{\lambda}{\sqrt{2}}\left(\tfrac{2}{3} - \tfrac{2}{5} + \tfrac{1}{14}\right)\right)$

$= a\exp\left(\tfrac{71}{210}\tfrac{\lambda}{\sqrt{2}}\right)$

• $y(\tfrac{1}{2}) = 0 \quad \Rightarrow \quad A\cos\left(\tfrac{239}{420}\tfrac{\lambda}{\sqrt{2}}\right) + B\sin\left(\tfrac{239}{420}\tfrac{\lambda}{\sqrt{2}}\right) = 0$

$\Rightarrow \quad A = -B\tan\left(\tfrac{239}{420}\tfrac{\lambda}{\sqrt{2}}\right)$

<u>Matching</u>:

• on the right of BL: as $x \to 0$,

$$y^{outer}(x) \simeq \frac{1}{x^{1/4}}\left\{A\cos\left(\tfrac{2\lambda}{3}x^{3/2}\right) + B\sin\left(x^{3/2}\cdot\tfrac{2\lambda}{3}\right)\right\}$$

But if $x = s\lambda^{-2/3}$ then this is

$$\simeq \frac{1}{s^{1/4}\lambda^{-1/6}}\left\{A\cos\left(\tfrac{2}{3}s^{3/2}\right) + B\sin\left(\tfrac{2}{3}s^{3/2}\right)\right\}$$

Matching to limit of $y^{inner}(s)$ as $s \to +\infty$,

$$\lim_{s\to+\infty} y(s) = \alpha\left[\frac{1}{\sqrt{\pi}\,s^{1/4}}\sin\left(\tfrac{2}{3}s^{3/2} + \tfrac{\pi}{4}\right)\right] + \beta\left[\frac{1}{\sqrt{\pi}\,s^{1/4}}\cos\left(\tfrac{2}{3}s^{\frac{3}{2}} + \tfrac{\pi}{4}\right)\right]$$

$$= \frac{\alpha}{\sqrt{\pi}\, s^{1/4}} \left[\sin\left(\tfrac{2}{3}s^{3/2}\right)\cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}\cos\left(\tfrac{2}{3}s^{3/2}\right)\right]$$

$$+ \frac{\beta}{\sqrt{\pi}\, s^{1/4}} \left[\cos\left(\tfrac{2}{3}s^{3/2}\right)\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\sin\left(\tfrac{2}{3}s^{3/2}\right)\right]$$

$$= \frac{\sqrt{2}}{2\sqrt{\pi}\, s^{1/4}} \left\{ \cos\left(\tfrac{2}{3}s^{3/2}\right)(\alpha+\beta) + \sin\left(\tfrac{2}{3}s^{3/2}\right)(\alpha-\beta)\right]$$

so we get $\dfrac{A}{\lambda^{-1/6}} = \dfrac{\sqrt{2}(\alpha+\beta)}{2\sqrt{\pi}}$ and $\dfrac{B}{\lambda^{-1/6}} = \dfrac{\sqrt{2}}{2\sqrt{\pi}}(\alpha-\beta)$.

- on the left of the BL, as $x \to 0$:

$$y^{outer}(x) \simeq \frac{1}{|x|^{1/4}} \left\{ a\exp\left(\tfrac{2}{3}|x|^{3/2}\lambda\right) + b\exp\left(-\tfrac{2}{3}\lambda|x|^{3/2}\right)\right\}$$

$$\simeq \frac{1}{|s|^{1/4}\lambda^{-1/6}} \left\{ a\exp\left(\tfrac{2}{3}|s|^{3/2}\right) + b\exp\left(-\tfrac{2}{3}|s|^{3/2}\right)\right\}$$

$$\lim_{s\to-\infty} y^{inner}(s)$$

$$= \frac{\alpha}{2\sqrt{\pi}\,|s|^{1/4}} \exp\left(-\tfrac{2}{3}|s|^{3/2}\right) + \frac{\beta}{\sqrt{\pi}\,|s|^{1/4}}\exp\left(\tfrac{2}{3}|s|^{3/2}\right)$$

so matching the two we get:

$$\frac{a}{\lambda^{-1/6}} = \frac{\beta}{\sqrt{\pi}} \quad \text{and} \quad \frac{b}{\lambda^{-1/6}} = \frac{\alpha}{2\sqrt{\pi}}$$

<u>Putting it all together:</u>

- eliminating b: $\quad a\exp\left(\frac{71}{210}\frac{\lambda}{\sqrt{2}}\right)\lambda^{1/6} = \frac{\alpha}{2\sqrt{\pi}}$
- eliminating A: $\quad -B\tan\left(\frac{239}{420}\frac{\lambda}{\sqrt{2}}\right)\lambda^{1/6} = \frac{\sqrt{2}}{2\sqrt{\pi}}(\alpha+\beta)$
- eliminating B: $\quad -\frac{\cancel{\sqrt{2}}}{\cancel{2}\sqrt{\pi}}(\alpha-\beta)\tan\left(\frac{239}{420}\frac{\lambda}{\sqrt{2}}\right) = \frac{\cancel{2}}{\cancel{2}\sqrt{\pi}}(\alpha+\beta)$

$$\Rightarrow \tan\left(\frac{239}{420}\frac{\lambda}{\sqrt{2}}\right) = \frac{\alpha+\beta}{\beta-\alpha}.$$

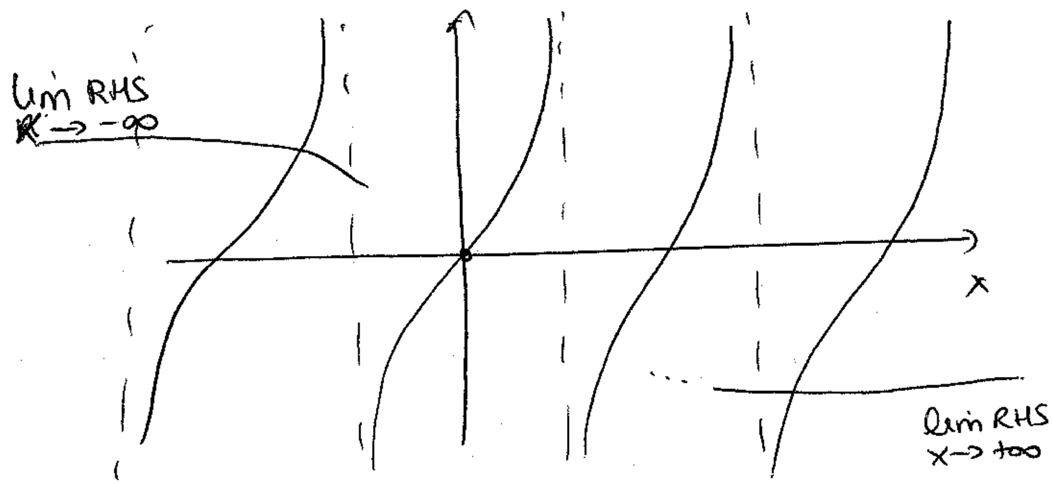
- eliminating a: $\quad \frac{\beta}{\cancel{\sqrt{\pi}}}\exp\left(\frac{71}{210}\frac{\lambda}{\sqrt{2}}\right) = \frac{\alpha}{2\cancel{\sqrt{\pi}}}.$

so finally: $\alpha = 2\beta \exp\left(\frac{71}{210}\frac{\lambda}{\sqrt{2}}\right)$ and

$$\tan\left(\frac{239}{420}\frac{\lambda}{\sqrt{2}}\right) = \frac{2\exp\left(\frac{71}{210}\frac{\lambda}{\sqrt{2}}\right)+1}{1-2\exp\left(\frac{71}{210}\frac{\lambda}{\sqrt{2}}\right)}.$$

If $x = \frac{71}{210}\frac{\lambda}{\sqrt{2}}$ then this is

$$\tan\left(\frac{239}{420}\frac{\lambda}{\sqrt{2}}\right) = \frac{2\exp(x)+1}{1-2\exp(x)}$$



so: as $x \to +\infty$ $\quad x \to -\frac{\pi}{4}+n\pi$ so

$$\frac{\lambda}{\sqrt{2}}\frac{239}{420} \to -\frac{\pi}{4}+n\pi$$

$$\to \lambda_n \to \left(-\frac{\pi}{4}+n\pi\right)\frac{420}{239}\sqrt{2}.$$

② Case $y(0) = 0$, $y(1) = 0$

This time we only have 1 outer, and one bl at boundary (at $x=0$). The inner solution is the same:

- BC @ $x=1$: $y(1) = 0 \Rightarrow$

$$A \cos\left(\lambda \cdot \left(\frac{2}{3}+\frac{4}{5}+\frac{2}{7}\right)\right) + B \sin\left(\lambda\left(\frac{2}{3}+\frac{4}{5}+\frac{2}{7}\right)\right) = 0$$

$$\Rightarrow \quad A = -B \tan\left(\frac{184}{105}\lambda\right)$$

- BC@ $x=0$: this is at $s=0$ in the inner solution so we need

$$y^{inner}(0) = 0 \Rightarrow \alpha Ai(0) + \beta Bi(0) = 0$$

$$\Rightarrow \frac{\alpha}{3^{\frac{2}{3}}\cancel{\Gamma(\frac{2}{3})}} + \frac{\beta}{3^{1/6}\cancel{\Gamma(\frac{2}{3})}} = 0$$

$$\Rightarrow \alpha = -\beta \frac{3^{\frac{2}{3}}}{3^{1/6}} = -\beta\, 3^{\frac{1}{2}} = -\sqrt{3}\beta$$

- Matching outer to inner: as in case 1, we get:

$$\frac{A}{\lambda^{1/6}} = \frac{\sqrt{2}}{2\sqrt{\pi}}(\alpha+\beta) \quad \text{and} \quad \frac{B}{\lambda^{-1/6}} = \frac{\sqrt{2}}{2\sqrt{\pi}}(\alpha-\beta)$$

Putting it all together we get:

- eliminating A: $$\frac{-B\tan\left(\frac{184}{105}\lambda\right)}{\lambda^{-1/6}} = \frac{\sqrt{2}}{2\sqrt{\pi}}(\alpha+\beta)$$

- eliminating α: $$= \frac{\sqrt{2}}{2\sqrt{\pi}}\left(\beta(1-\sqrt{3})\right)$$

- eliminating B: $$-\tan\left(\frac{184}{105}\lambda\right)\cdot \cancel{\frac{\sqrt{2}}{2\sqrt{\pi}}}(-\sqrt{3}-1)\cancel{\beta} = \cancel{\frac{\sqrt{2}}{2\sqrt{\pi}}}\cancel{\beta}(1-\sqrt{3})$$

$$\Rightarrow \tan\left(\frac{184}{105}\lambda\right) = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$