HOMEWORK 7

Problem 1

(ii) $\frac{d^2y}{dx^2} + (1+x)y = 0$

let $y = \exp(\phi_0(x) + \phi_1(x) + \cdots)$ where $\{\phi_n(x)\}$ form an asymptotic sequence as $x \to +\infty$

(1) $\Rightarrow$ To lowest order:

$$\frac{d^2\phi_0}{dx^2} + \left(\frac{d\phi_0}{dx}\right)^2 + (1+x) = 0$$

$$\to \quad \frac{d^2\phi_0}{dx^2} + \left(\frac{d\phi_0}{dx}\right)^2 + x = 0$$

Case 1: $\frac{d^2\phi_0}{dx^2} + x = 0 \Rightarrow \frac{d\phi_0}{dx} = -\frac{x^2}{2} + C$

$$\Rightarrow \phi_0(x) = -\frac{x^3}{3} + Cx + D$$

Inconsistent: $\left(\frac{d\phi_0}{dx}\right)^2$ not small compared with kept term x

Case 2: $\frac{d^2\phi_0}{dx^2} + \left(\frac{d\phi_0}{dx}\right)^2 = 0 \Rightarrow \phi_0'^2(x) = \frac{1}{(x-c)^2}$ (see lectures)

Inconsistent $\to$ this is much smaller than neglected term x.

Case 3: $\left(\frac{d\phi_0}{dx}\right)^2 + x = 0 \Rightarrow \frac{d\phi_0}{dx} = \pm i\sqrt{x}$

$$\Rightarrow \phi_0(x) = \pm \frac{2}{3} i x^{3/2} + c.$$

Check: $\frac{d^2\phi_0}{dx^2} = \pm \frac{i}{2\sqrt{x}} \to$ is small compared with kept terms ✓

(2) To next order:

$$\underbrace{\frac{d^2\phi_0}{dx^2}} + 2\frac{d\phi_0}{dx}\frac{d\phi_1}{dx} + 1 = 0$$

this is actually still negligible compared with $1 \Rightarrow$

2.

$$\frac{d\phi_1}{dx} = -\frac{1}{2\frac{d\phi_0}{dx}} = \mp\frac{1}{2i\sqrt{x}} \Rightarrow \phi_1(x) = \pm i\sqrt{x} + \text{const}$$

③ To next order

$$\frac{d^2\phi_0}{dx^2} + \left(\frac{d\phi_1}{dx}\right)^2 + 2\frac{d\phi_0}{dx}\frac{d\phi_2}{dx} = 0$$

$$\Rightarrow \pm\frac{i}{2\sqrt{x}} + \underbrace{\left(\frac{1}{2i\sqrt{x}}\right)^2}_{\text{negligible}} + 2\frac{d\phi_0}{dx}\frac{d\phi_2}{dx} = 0$$

$$\rightarrow \frac{d\phi_2}{dx} = \mp\frac{i}{4\sqrt{x}\frac{d\phi_0}{dx}} = \mp\frac{\cancel{i}}{4\sqrt{x}\cdot(\pm\cancel{i}\sqrt{x})}$$

$$= -\frac{1}{4x} \rightarrow \phi_2(x) = -\frac{1}{4}\ln x + \text{const.}$$

$$= \ln\left(x^{-\frac{1}{4}}\right) + \text{const}$$

④ To next order:

$$\frac{d^2\phi_1}{dx^2} + \left(\frac{d\phi_1}{dx}\right)^2 + 2\frac{d\phi_0}{dx}\frac{d\phi_3}{dx} + \underbrace{2\frac{d\phi_1}{dx}\frac{d\phi_2}{dx}}_{\text{negligible compared with } \left(\frac{d\phi_1}{dx}\right)^2} = 0$$

$$\rightarrow \underbrace{\pm\frac{1}{4i}x^{-3/2}}_{\text{negligible compared with } \left(\frac{d\phi_1}{dx}\right)^2} + \left(\frac{1}{2i\sqrt{x}}\right)^2 + \frac{2d\phi_0}{dx}\frac{d\phi_3}{dx} = 0$$

$$\frac{d\phi_3}{dx} = \frac{1}{4x}\cdot\left(\frac{1}{\pm 2i\sqrt{x}}\right)$$

$$\rightarrow \phi_3(x) = O\left(x^{-1/2}\right)$$

So finally,

$$y(x) = \exp\left[\pm\frac{2}{3}ix^{3/2} \pm i\sqrt{x} + \ln\left(x^{-\frac{1}{4}}\right) + O\left(x^{-1/2}\right) + \text{const}\right]$$

$$= \frac{A}{x^{1/4}}\cos\left(\frac{2}{3}x^{3/2} + x^{1/2} + O\left(x^{-1/2}\right)\right)$$

$$+ \frac{B}{x^{1/4}}\sin\left(\frac{2}{3}x^{3/2} + x^{1/2} + O\left(x^{-1/2}\right)\right)$$

(iv) $\frac{d^2y}{dx^2} - x^2 y = 0$

let $y(x) = \exp\left(\phi_0(x) + \phi_1(x) + \cdots\right)$ where $\phi_n(x)$ forms an asymptotic sequence as $x \to +\infty$

① <u>To lowest order:</u>

$$\frac{d^2\phi_0}{dx^2} + \left(\frac{d\phi_0}{dx}\right)^2 - x^2 = 0$$

$\to \left(\frac{d\phi_0}{dx}\right)^2 = x^2$ is dominant balance

$\to \frac{d\phi_0}{dx} = \pm x \to \phi_0(x) = \pm \frac{x^2}{2} + c$

<u>Note:</u> $\frac{d^2\phi_0}{dx^2} = \pm 1 \Rightarrow$ indeed negligible.

② <u>To next order.</u>

$$\frac{d^2\phi_0}{dx^2} + 2\frac{d\phi_0}{dx}\frac{d\phi_1}{dx} = 0$$

$$\Rightarrow \frac{d\phi_1}{dx} = -\frac{\frac{d^2\phi_0}{dx^2}}{2\frac{d\phi_0}{dx}} = -\frac{\pm 1}{\pm 2x} = -\frac{1}{2x}$$

$$\Rightarrow \phi_1(x) = -\frac{1}{2}\ln x + \text{const.}$$

$$= \ln(x^{-\frac{1}{2}}) + \text{const.}$$

③ <u>To next order</u>

$$\frac{d^2\phi_1}{dx^2} + \left(\frac{d\phi_1}{dx}\right)^2 + 2\frac{d\phi_0}{dx}\frac{d\phi_2}{dx} = 0$$

$$\frac{1}{2x^2} + \left(\frac{1}{2x}\right)^2 + 2\frac{d\phi_0}{dx}\frac{d\phi_2}{dx} = 0$$

$$\Rightarrow \frac{d\phi_2}{dx} = \frac{-\frac{3}{4x^2}}{\pm 2x} = \mp \frac{3}{8x^3}$$

$$\Rightarrow \phi_2(x) = \pm \frac{3}{16x^2}$$ $\to$ goes to 0, we can stop here.

$$\to y(x) = \exp\left(\pm \frac{x^2}{2} + \ln(x^{-\frac{1}{2}}) \pm \frac{3}{16x^2} + \text{const} + \text{h.o.t}\right)$$

$$\Rightarrow \quad y(x) = \frac{A}{\sqrt{x}} \exp\left(\frac{x^2}{2} + \frac{3}{16x^2} + \text{h.o.t}\right) + \frac{B}{\sqrt{x}} \exp\left(-\frac{x^2}{2} - \frac{3}{16x^2} + \text{h.o.t}\right).$$

(vi) $\frac{d^2y}{dx^2} - (2+x^2)y = 0$

as usual let $y = \exp(\phi_0(x) + \phi_1(x) + \cdots)$

$\rightarrow$ ① To lowest order

$$\frac{d^2\phi_0}{dx^2} + \left(\frac{d\phi_0}{dx}\right)^2 - (2+x^2) = 0$$

$\rightarrow$ dominant balance is

$$\left(\frac{d\phi_0}{dx}\right)^2 - x^2 = 0 \Rightarrow \frac{d\phi_0}{dx} = \pm x$$

$$\Rightarrow \phi_0(x) = \pm \frac{x^2}{2} + C$$

② To next order

$$\frac{d^2\phi_0}{dx^2} + 2\frac{d\phi_0}{dx}\frac{d\phi_1}{dx} - 2 = 0$$

$$\pm 1 + 2\frac{d\phi_0}{dx}\frac{d\phi_1}{dx} - 2 = 0$$

$$\rightarrow \frac{d\phi_1}{dx} = \frac{2 \mp 1}{\pm 2x} = \begin{cases} \frac{1}{2x} \\ -\frac{3}{2x} \end{cases}$$

$$\rightarrow \phi_1(x) = \begin{cases} \frac{1}{2}\ln x + \text{const.} \\ -\frac{3}{2}\ln x + \text{const} \end{cases}$$

③ To next order

$$\frac{d^2\phi_1}{dx^2} + \left(\frac{d\phi_1}{dx}\right)^2 + 2\frac{d\phi_0}{dx}\frac{d\phi_2}{dx} = 0$$

$$\begin{pmatrix} -\frac{1}{2x^2} \\ \frac{3}{2x^2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4x^2} \\ \frac{9}{4x^2} \end{pmatrix} + 2\cdot(\pm x)\cdot\frac{d\phi_2}{dx} = 0.$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{4x^2} \\ \frac{15}{4x^2} \end{pmatrix} \pm 2x \frac{d\phi_2}{dx} = 0$$

$$\Rightarrow \frac{d\phi_2}{dx} = \mp \frac{1}{2x} \begin{pmatrix} -\frac{1}{4x^2} \\ \frac{15}{4x^2} \end{pmatrix} = \begin{pmatrix} +\frac{1}{8x^3} \\ +\frac{15}{8x^3} \end{pmatrix}$$

$$\rightarrow \phi_2(x) = \begin{pmatrix} -\frac{1}{16x^2} \\ -\frac{15}{16x^2} \end{pmatrix} \leftarrow \text{goes to 0 so we can stop here}$$

$$\Rightarrow y(x) = A\exp\left(\frac{x^2}{2} + \frac{1}{2}\ln x - \frac{1}{16x^2} + \text{const} \right) + B\exp\left(-\frac{x^2}{2} - \frac{3}{2}\ln x - \frac{15}{16x^2} \right)$$

$$= A\sqrt{x}\, \exp\left(\frac{x^2}{2} - \frac{1}{16x^2} \right) + \frac{B}{x^{3/2}} \exp\left(-\frac{x^2}{2} - \frac{15}{16x^2} \right)$$

Problem 2 :

$$\frac{d^2y}{dx^2} + (\lambda^2x^2 + x)y = 0$$

→ Not quite in Liouville form so we have to do it the long way

① Case $x > 0$

let $y(x) = \exp\left(\phi_0(x)\delta_0(\lambda) + \phi_1(x)\delta_1(\lambda) + \cdots\right)$

- to lowest order:

$$\frac{d^2\phi_0}{dx^2}\delta_0(\lambda) + \left(\frac{d\phi_0}{dx}\right)^2\delta_0^2(\lambda) + \lambda^2x^2 + \underbrace{x}_{\text{negligible}} = 0$$

Case 1 : $\frac{d^2\phi_0}{dx^2}\delta_0(\lambda) + \lambda^2x^2 = 0$

$\rightarrow \delta_0(\lambda) = \lambda^2 \rightarrow \delta_0(\lambda) = \sqrt{\lambda}$

so neglected term would be $O(\lambda^4)$ → inconsiste

Case 2 : $\frac{d^2\phi_0}{dx^2}\delta_0(\lambda) + \left(\frac{d\phi_0}{dx}\right)^2\delta_0^2(\lambda) = 0$

$\rightarrow \delta_0(\lambda) = 1$ so neglected term is $O(\lambda^2)$

→ inconsistent

Case 3 : $\left(\frac{d\phi_0}{dx}\right)^2\delta_0^2(\lambda) + \lambda^2x^2 = 0$

$\rightarrow \delta_0(\lambda) = \lambda$ → neglected term is $O(\lambda)$

while kept terms are $O(\lambda^2)$ ✓

$$\rightarrow \left(\frac{d\phi_0}{dx}\right)^2 = -x^2 \Rightarrow \frac{d\phi_0}{dx} = \pm ix$$

$$\Rightarrow \phi_0(x) = \pm i\frac{x^2}{2} + \text{const}$$

- To next order:

$$\left(\frac{d^2\phi_0}{dx^2}\right)\delta_0(\lambda) + 2\frac{d\phi_0}{dx}\frac{d\phi_1}{dx}\delta_0(\lambda)\delta_1(\lambda) + x = 0$$

$$\rightarrow \pm i\lambda + (\pm 2ix)\frac{d\phi_1}{dx}\lambda\delta_1(\lambda) + \underbrace{x}_{\text{negligible}} = 0$$

$$\rightarrow \begin{cases} \delta_1(\lambda) = 1 \\ d\phi_1/dx = -\frac{1}{2x} \end{cases} \rightarrow \phi_1(x) = -\frac{1}{2}\ln x + \text{con}$$

- To next order

$$\left(\frac{d^2\phi_1}{dx^2}\right)\delta_1(\lambda) + \left(\frac{d\phi_1}{dx}\right)^2 \delta_1^2(\lambda) + 2\frac{d\phi_0}{dx}\frac{d\phi_2}{dx}\delta_0(\lambda)\delta_2(\lambda) + x = 0$$

$$\rightarrow \quad \frac{1}{2x^2} + \left(\frac{1}{4x^2}\right) \pm 2ix\frac{d\phi_2}{dx}\lambda\,\delta_2(\lambda) + x = 0$$

$$\Rightarrow \begin{cases} \dfrac{d\phi_2}{dx} = \pm\left[-\dfrac{x + \frac{3}{4x^2}}{2ix}\right] = \pm i\left(\dfrac{1}{2} + \dfrac{3}{8x^3}\right) \\ \delta_2(\lambda) = \dfrac{1}{\lambda} \quad \text{and} \\ \phi_2 = \pm\dfrac{ix}{2} \mp \dfrac{3i}{16x^2} \end{cases}$$

so, finally,

$$y(x) = \exp\left[\pm i\lambda\frac{x^2}{2} - \frac{1}{2}\ln x \pm \frac{i}{2\lambda}\left(x - \frac{3}{8x^2}\right)\right]$$

$$= \frac{A}{x^{1/2}}\cos\left(\lambda\frac{x^2}{2} + \frac{1}{2\lambda}\left(x - \frac{3}{8x^2}\right)\right) + \frac{B}{x^{1/2}}\sin\left(\lambda\frac{x^2}{2} + \frac{1}{2\lambda}\left(x - \frac{3}{8x^2}\right)\right)$$

<u>Case $x<0$</u>

Following the same steps we get:

$$\phi_0(x) = \pm\frac{ix^2}{2} + \text{const} \qquad \delta_0 = \lambda$$

$$\phi_1(x) = -\frac{1}{2}\ln|x| \qquad \delta_1 = 1$$

$$\phi_2(x) = \pm\frac{ix}{2} \mp \frac{3i}{16x^2} \qquad \delta_2 = \frac{1}{\lambda}$$

$\rightarrow$ the solution $y(x)$ is

$$y(x) = \frac{A}{|x|^{1/2}}\cos\left(\lambda\frac{x^2}{2} + \frac{1}{2\lambda}\left(x - \frac{3}{8x^2}\right)\right) + \frac{B}{|x|^{1/2}}\sin\left(\lambda\frac{x^2}{2} + \frac{1}{2\lambda}\left(x - \frac{3}{8x^2}\right)\right)$$

(iv) $\frac{d^2y}{dx^2} + \lambda^2(x-1)\,y = 0$.

This problem is in Liouville form, so we can use the solution obtained in notes:

$\underline{x>1}$:

$$y(x) = \frac{1}{|x-1|^{1/4}}\left\{ A\cos\left(\lambda\int\sqrt{x'-1}\,dx' + o\left(\tfrac{1}{\lambda}\right)\right) + B\sin\left(\lambda\int\sqrt{x'-1}\,dx' + o\left(\tfrac{1}{\lambda}\right)\right)\right\}$$

$$= \frac{1}{(x-1)^{1/4}}\left\{ A\cos\left(\tfrac{2}{3}\lambda(x-1)^{3/2}\right) + B\sin\left(\tfrac{2}{3}\lambda(x-1)^{3/2}\right)\right\}$$

$\underline{x<1}$: similarly, with exponential functions:

$$y(x) = \frac{1}{(1-x)^{1/4}}\left\{ A\exp\left(\tfrac{2}{3}\lambda(1-x)^{3/2}\right) + B\exp\left(-\tfrac{2}{3}\lambda(1-x)^{3/2}\right)\right\}$$