HOMEWORK 4.

Problem 1 p 97

(1) $(x+\varepsilon y)\dfrac{dy}{dx} + 3y = x^2$

Let $y = y_0 + \varepsilon y_1 + \cdots$

→ to zeroth order

$$\begin{cases} x\dfrac{dy_0}{dx} + 3y_0 = x^2 \\ y_0(1) = 1 \end{cases}$$

$$\rightarrow \quad \frac{dy_0}{dx} + \frac{3y_0}{x} = x$$

We solve this using an integrating factor:

$$\mu(x) = e^{\int\frac{3}{x}} = e^{3\ln x} = x^3 \Rightarrow$$

$$\frac{d}{dx}\left(x^3 y_0\right) = x^4 \Rightarrow x^3 y_0(x) - y_0(1) = \int_1^x x'^4\,dx'$$

$$\Rightarrow \quad x^3 y_0(x) = 1 + \left(\frac{x^5}{5} - 1\right) = \frac{x^5}{5} - \frac{1}{5} + 1$$

so $\quad y_0(x) = \dfrac{x^2}{5} + \dfrac{4}{5x^3}$

To first order

$$\begin{cases} x\dfrac{dy_1}{dx} + 3y_1 + y_0\dfrac{dy_0}{dx} = 0 \\ y_1(1) = 0 \end{cases}$$

$$\Rightarrow \quad x\frac{dy_1}{dx} + 3y_1 = -\left(\frac{x^2}{5} + \frac{4}{5x^3}\right)\left(\frac{2x}{5} - \frac{12}{5x^4}\right)$$

$$= -\frac{2x^3}{25} + \frac{12}{25x^2} - \frac{8}{25x^2} + \frac{48}{25x^7}$$

$$= -\frac{1}{25}\left[2x^3 - \frac{4}{x^2} - \frac{48}{x^7}\right]$$

This has the same integrating factor as y_0 so

$$\frac{d}{dx}(x^3 y_1) = -\frac{1}{25}\left[2x^5 - 4 - \frac{48}{x^5}\right]$$

$$\Rightarrow \quad x^3 y_1(x) - 1^3 y_1(0) = \int_1^x -\frac{1}{25}\left(2x'^5 - 4 - \frac{48}{x'^5}\right)dx'$$

$$= -\frac{2}{25}\cdot\frac{1}{6}\left(x^6 - 1\right) + \frac{4}{25}(x-1) + \frac{48}{25}\left(-\frac{1}{4}\right)\left(\frac{1}{x^4} - 1\right)$$

$$\Rightarrow \quad y_1(x) = -\frac{1}{75}\left(x^3 - \frac{1}{x^3}\right) + \frac{4}{25}\left(\frac{1}{x^2} - \frac{1}{x^3}\right) - \frac{12}{25}\left(\frac{1}{x^7} - \frac{1}{x^3}\right)$$

and so

$$y(x) = \frac{x^2}{5} + \frac{4}{5x^3} + \varepsilon\left[\frac{4}{25x^2} + \frac{1}{3x^3} - \frac{12}{25x^7} - \frac{x^3}{75}\right].$$

To get a uniform expansion as $x \to 0$, let

$$x = s + \varepsilon f_1(s) + \cdots$$

then

$$y(s) = \frac{(s+\varepsilon f_1 + \cdots)^2}{5} + \frac{4}{5(s+\varepsilon f_1 \cdots)^3} + \varepsilon\left[\frac{4}{25s^2} + \frac{1}{3s^3} - \frac{12}{25s^7} - \frac{s^3}{75}\right]$$

$$= \frac{s^2}{5} + \frac{2sf_1\varepsilon}{5} + \frac{4}{5s^3} - \frac{12 f_1 \varepsilon}{5s^4} \cdots + \varepsilon[\;\cdots\;]$$

If we want to eliminate all terms in the $[\ldots]$ that go to 0 faster than $\frac{1}{s^3}$ as $s \to 0$, we need to let

$$-\frac{12 f_1}{5s^4} = \frac{12}{25s^7} \Rightarrow \boxed{f_1 = -\frac{1}{5s^3}}$$

This then implies $x = s - \frac{\varepsilon}{5s^3} + \cdots$ as a possible expansion for x

So

$$\begin{cases} f(s) = \dfrac{s^2}{5} + \dfrac{4}{5s^3} + O(\varepsilon)\cdots \\ x = s - \dfrac{\varepsilon}{5s^3} + O(\varepsilon^2). \end{cases}$$

This look a bit suspicious, but note that as $x \to 0$, one of the solutions of

$$s - \frac{\varepsilon}{5s^3} = 0 \quad \text{is} \quad s^4 = \frac{\varepsilon}{5} \to s = \left(\frac{\varepsilon}{5}\right)^{1/4},$$

and is therefore non-singular.

This implies that $\lim_{x \to 0} f(x) = \lim_{s \to \left(\frac{\varepsilon}{5}\right)^{1/4}} f(s) \cong \frac{4}{5}\left(\frac{\varepsilon}{5}\right)^{-3/4} + \text{h.o.t.}$

$\to$ the limit is finite.

(2) $(x+\varepsilon y)\frac{dy}{dx} + 2y = 1 \qquad y(1) = 2$

- First let's find the standard expansion of this equation:

let $y = y_0 + \varepsilon y_1 + \cdots$

$$\to \quad (x + \varepsilon y_0 + \varepsilon^2 y_1 + \cdots)(y_0' + \varepsilon y_1' + \cdots) + 2(y_0 + \varepsilon y_1 + \cdots) = 1$$

$$\begin{cases} y_0(1) = 2 \\ y_1(1) = 0 \end{cases}$$

To lowest order,

$$x y_0' + 2y_0 = 1 \quad \to \quad y_0' + \frac{2}{x} y_0 = \frac{1}{x}$$

Integrating factor: $\mu(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

so let $\frac{d}{dx}(x^2 y_0) = x \Rightarrow x^2 y_0 = \frac{x^2}{2} + k$

$$\Rightarrow \quad y_0(x) = \frac{1}{2} + \frac{k}{x^2}$$

If $y_0(1) = 2 \Rightarrow 2 = \frac{1}{2} + k \Rightarrow k = \frac{3}{2}$

So $y_0(x) = \frac{1}{2} + \frac{3}{2x^2}$

To first order,

$$x y_1' + y_0 y_0' + 2y_1 = 0$$

$$\Rightarrow \quad x y_1' + 2y_1 = -\left(\frac{1}{2} + \frac{3}{2x^2}\right)\left(-\frac{3}{x^3}\right) = +\frac{3}{2x^3} + \frac{9}{2x^5}$$

This has the same integrating factor as before,

so
$$\frac{d}{dx}\left(x^2 y_1\right) = +\frac{3}{2x^2} + \frac{9}{2x^4}$$

$$x^2 y_1 - K = -\frac{3}{2x} - \frac{3}{2x^3}$$

$$\Rightarrow y_1 = \frac{K}{x^2} - \frac{3}{2x^3} - \frac{3}{2x^5}$$

$$y_1(1) = 0 \Rightarrow 0 = K - \frac{3}{2} - \frac{3}{2} \text{ so } K = 3$$

$$\Rightarrow y_1(x) = \frac{3}{x^2} - \frac{3}{2x^3} - \frac{3}{2x^5}$$

So finally $y(x) = \frac{1}{2} + \frac{3}{2x^2} - \varepsilon\left[-\frac{3}{x^2} + \frac{3}{2x^3} + \frac{3}{2x^5}\right]$

to have a uniform expansion as $x \to 0$, we'd like to get rid of the $\frac{3}{2x^3}$ and $\frac{3}{2x^5}$ terms.

So let $x = s + a_1(s)\varepsilon + \cdots$

$$\Rightarrow y(s) = \frac{1}{2} + \frac{3}{2(s + a_1(s)\varepsilon)^2} - \varepsilon\left[-\frac{3}{s^2} + \frac{3}{2s^3} + \frac{3}{2s^5}\right] + \cdots$$

$$= \frac{1}{2} + \frac{3}{2s^2}\left(1 - \frac{2a_1(s)\varepsilon}{s}\right) - \varepsilon\left[-\frac{3}{s^2} + \frac{3}{2s^3} + \frac{3}{2s^5}\right]$$

So we want
$$-\frac{6a_1(s)}{2s^3} = \frac{3}{2s^3} + \frac{3}{2s^5}$$

$$\Rightarrow a_1(s) = \frac{-1}{2}\left(1 + \frac{1}{s^2}\right)$$

and so $x = s - \frac{\varepsilon}{2}\left(1 + \frac{1}{s^2}\right)$

$\Rightarrow$ our uniform expansion is
$$\begin{cases} x = s - \frac{\varepsilon}{2}\left(1 + \frac{1}{s^2}\right) \\ y(s) = \frac{1}{2} + \frac{3}{2s^2} + O(\varepsilon) \end{cases}$$

Finally, note that when $x=0$

$$s = \frac{\varepsilon}{4}\left(1+\frac{1}{s^2}\right)$$

This has a regular solution (when s is small) with dominant balance $s = \frac{\varepsilon}{4s^2} \Rightarrow s^3 = \frac{\varepsilon}{4}$

$\rightarrow \quad s = \left(\frac{\varepsilon}{2}\right)^{1/3}$

So $\quad y(x\to 0) = \frac{1}{2} + \frac{3}{2\left(\frac{\varepsilon}{2}\right)^{2/3}}$

(3) $\quad (x+\varepsilon y)\frac{dy}{dx} + 3y = 2 \qquad y(1)=1$

As before

Standard expansion: let $y = y_0 + \varepsilon y_1 + \cdots$

$$(x+\varepsilon(y_0+\varepsilon y_1+\cdots))(y_0'+\varepsilon y_1'+\cdots) + 3(y_0+\varepsilon y_1+\cdots) = 2$$

$\Rightarrow$ to 0^{th} order $\begin{cases} xy_0' + 3y_0 = 2 \\ y_0(1) = 1 \end{cases}$

$\Rightarrow \quad y_0' + \frac{3}{x}y_0 = \frac{2}{x} \qquad \mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$

$\Rightarrow \quad \frac{d}{dx}(x^3 y_0) = 2x^2 \quad \Rightarrow \quad x^3 y_0 = \frac{2x^3}{3} + K$

at $y_0(1)=1 \Rightarrow \quad 1 = \frac{2}{3} + K$

so $\quad K = \frac{1}{3}$

$\Rightarrow \quad y_0(x) = \frac{2}{3} + \frac{1}{3x^3}$

To 1st order: $\begin{cases} xy_1' + 3y_1 + y_0 y_0' = 0 \\ y_1(1) = 0 \end{cases}$

$\Rightarrow \quad \frac{d}{dx}(x^3 y_1) = -x^2 y_0 y_0'$

$$= -x^2\left(\frac{2}{3}+\frac{1}{3x^3}\right)\left(-\frac{1}{x^4}\right) = \frac{2}{3x^2} + \frac{1}{3x^5}$$

$$\Rightarrow x^3 y_1 - K = -\frac{2}{3x} - \frac{1}{12x^4}$$

$$\Rightarrow \text{at } y_1(1) = 0 \quad \Rightarrow \quad -K = -\frac{2}{3} - \frac{1}{12} = -$$

$$\Rightarrow K = \frac{3}{4}$$

So $y_1(x) = \frac{3}{4x^3} - \frac{2}{3x^4} - \frac{1}{12x^7}$

$$\Rightarrow y(x) = \frac{2}{3} + \frac{1}{3x^3} + \varepsilon\left[\frac{3}{4x^3} - \frac{2}{3x^4} - \frac{1}{12x^7}\right] + hot$$

We want to elimenate the terms in $\frac{1}{x^4}$ and $\frac{1}{x^7}$

→ let's assume

$$x = s + \varepsilon a_1(s) + \cdots$$

$$\Rightarrow y(s) = \frac{2}{3} + \frac{1}{3(s+\varepsilon a_1(s))^3} + \varepsilon\left[\frac{3}{4s^3} - \frac{2}{3s^4} - \frac{1}{12s^7}\right] + hot$$

$$= \frac{2}{3} + \frac{1}{3s^3}\left(1 - \frac{3\varepsilon a_1}{s}\right) + \varepsilon[\cdots]$$

to eliminate the unwanted terms we simply need

$$\frac{a_1}{s^4} = -\frac{2}{3s^4} - \frac{1}{12s^7}$$

$$\Rightarrow a_1 = -\left(\frac{2}{3} + \frac{1}{12s^3}\right)$$

So the uniform expansion is

$$\begin{cases} x = s - \varepsilon\left(\frac{2}{3} + \frac{1}{12s^3}\right) \\ y(s) = \frac{2}{3} + \frac{1}{3s^3} + O(\varepsilon) \end{cases}$$

as $x \to 0$ we have $s \simeq \varepsilon\left(\frac{2}{3} + \frac{1}{12s^3}\right)$

this has the approximate solution $s^4 = \frac{\varepsilon}{12}$

$$\rightarrow s = \left(\frac{\varepsilon}{12}\right)^{1/4}$$

$$\Rightarrow y(x \to 0) = \frac{2}{3} + \frac{1}{3}\left(\frac{\varepsilon}{12}\right)^{-3/4}$$

(4) $(x+\varepsilon y)\frac{dy}{dx} + 4y = 0 \qquad y(1)=1$

Again, as before:

$y = y_0 + \varepsilon y_1 + \cdots$

$$\Rightarrow \quad x\frac{dy_0}{dx} + 4y_0 = 0 \;, \quad y_0(1)=1$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^4 y_0\right) = 0 \quad \Rightarrow \quad x^4 y_0 = K \Rightarrow y_0 = \frac{K}{x^4}$$

$$y_0(1) = 1 \quad \Rightarrow \quad K = 1 \quad \text{and so } y_0(x) = \frac{1}{x^4}$$

$$x\frac{dy_1}{dx} + 4y_1 + y_0\frac{dy_0}{dx} = 0 \qquad y_1(1) = 0$$

$$\Rightarrow \quad x\frac{dy_1}{dx} + 4y_1 = -\frac{1}{x^4}\left(-\frac{4}{x^5}\right) = \frac{4}{x^9}$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^4 y_1\right) = \frac{4}{x^6}$$

$$\Rightarrow \quad x^4 y_1 - K = -\frac{4}{5x^5}$$

$$\Rightarrow \quad y_1 = \frac{K}{x^4} - \frac{4}{5x^9}$$

$$y_1(1) = 0 \quad \Rightarrow \quad 0 = K - \frac{4}{5} \quad \Rightarrow K = \frac{4}{5}$$

$$\text{so } y_1(x) = \frac{4}{5x^4} - \frac{4}{5x^9}$$

$$\text{so } y(x) = \frac{1}{x^4} + \frac{4\varepsilon}{5}\left(\frac{1}{x^4} - \frac{1}{x^9}\right)$$

We want to get rid of the $\frac{1}{x^9}$ term

let $x = s + \varepsilon a_1(s) + \cdots \quad \Rightarrow$

$$y(s) = \frac{1}{(s+\varepsilon a_1(s))^4} + \frac{4\varepsilon}{5}\left(\frac{1}{s^4} - \frac{1}{s^9}\right)$$

$$= \frac{1}{s^4}\left(1 - \frac{4\varepsilon a_1}{s}\right) + \frac{4\varepsilon}{5}\left(\frac{1}{s^4} - \frac{1}{s^9}\right)$$

$$\Rightarrow \text{we need } \quad \frac{4a_1}{s^5} = -\frac{4}{5s^9} \quad \Rightarrow \quad a_1 = -\frac{1}{5s^4}.$$

$$\text{So finally, } \quad x = s - \frac{\varepsilon}{5s^4} + O(\varepsilon^2)$$

and the uniform expansion has

$$\begin{cases} x = s - \dfrac{\varepsilon}{5s^4} \\ y(s) = \dfrac{1}{s^4} + O(\varepsilon) \end{cases}$$

as $x \to 0$, $\quad s \simeq \dfrac{\varepsilon}{5s^4}$ so $\quad s \simeq \left(\dfrac{\varepsilon}{5}\right)^{1/5}$

so $y(x \to 0) \simeq \left(\dfrac{\varepsilon}{5}\right)^{-4/5}$.

Problem 2 page 131 (odd #s)

(1) $\dfrac{d^2u}{dt^2} + u = -\varepsilon\left(\dfrac{du}{dt}\right)^3 \qquad u(0) = a \qquad \dfrac{du}{dt}(0) = 0$

let $T_0 = t, \; T_1 = \varepsilon t$

$$u = u_0 + \varepsilon u_1 + \cdots$$

$$\left(\frac{\partial^2}{\partial T_0^2} + \varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2}\right)(u_0 + \varepsilon u_1 + \cdots) + (u_0 + \varepsilon u_1 + \cdots)$$
$$= -\varepsilon\left(\left(\frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}\right)(u_0 + \varepsilon u_1 + \cdots)\right)^3$$

To zeroth order:

$$\begin{cases} \dfrac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \\ u_0(0,0) = a \\ \dfrac{\partial u_0}{\partial T_0}(0,0) = 0 \end{cases} \Rightarrow u_0(T_0,T_1) = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0$$

with $A_0(0) = a$
$B_0(0) = 0$.

Alternatively: $u_0 = A_0 e^{iT_0} + A_0^* e^{-iT_0}$
with $\begin{cases} A_0 + A_0^*(0) = a \\ A_0 - A_0^*(0) = 0 \end{cases}$

To first order:

$$\begin{cases} 2\dfrac{\partial^2 u_0}{\partial T_0 \partial T_1} + \dfrac{\partial^2 u_1}{\partial T_0^2} + u_1 = -\left(\dfrac{\partial u_0}{\partial T_0}\right)^3 \\ \text{ICs irrelevant} \end{cases}$$

$$\Rightarrow \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -i\left(2\frac{\partial A_0}{\partial T_1}e^{iT_0} - 2\frac{\partial A_0^*}{\partial T_1}e^{-iT_0}\right) - \left[i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)\right]^3$$

$$= -i\left(2\frac{\partial A_0}{\partial T_1}e^{iT_0} - 2\frac{\partial A_0^*}{\partial T_1}e^{-iT_0}\right) + i\left(A_0^3 e^{3iT_0} - 3A_0^2 A_0^* e^{iT_0} + 3A_0 A_0^{*2} e^{-iT_0} - A_0^{*3} e^{-3iT_0}\right)$$

$\Rightarrow$ compatibility condition is:

$$-2i\frac{\partial A_0}{\partial T_1} - 3iA_0^2 A_0^* = 0$$

$$\Rightarrow \boxed{2\frac{dA_0}{dT_1} = -3A_0^2 A_0^*}$$

Let $A_0 = |A_0|e^{i\theta}$ then

$$2\frac{d|A_0|}{dT_1}e^{i\theta} + 2i\frac{d\theta}{dT_0}|A_0|e^{i\theta} = -3|A_0|^2|A_0|e^{i\theta}$$

$\Rightarrow$ can be solved provided

$$\begin{cases} 2\frac{d|A_0|}{dT_1} = -3|A_0|^3 \\ 2\frac{d\theta}{dT_0} = 0 \rightarrow \theta(T) = 0 \text{ always} \end{cases}$$

(using A_0 real @ $T_1 =$

$$\Rightarrow \frac{d|A_0|}{|A_0|^3} = -\frac{3dT_1}{2} \Rightarrow -\frac{\frac{1}{2}}{|A_0|^2} = -\frac{3T_1}{2} + C$$

So with $|A_0| = A_0 = \frac{a}{2}$ at $T_1 = 0$,

$$\Rightarrow -\frac{1}{2}\cdot\frac{1}{\frac{a^2}{4}} = C \Rightarrow C = -\frac{2}{a^2}$$

$$\Rightarrow -\frac{1}{2|A_0|^2} = -\frac{3T_1}{2} - \frac{2}{a^2}$$

So $|A_0|^2 = \dfrac{1}{3T_1 + \frac{4}{a^2}}$

$$\Rightarrow |A_0|^2 = \frac{1}{3T_1 + \frac{4}{a^2}} \Rightarrow |A_0| = \frac{1}{\sqrt{3T_1 + \frac{4}{a^2}}} = A_0$$

finally: $u(t) = u_0(T_0, T_1) = \dfrac{1}{\sqrt{3T_1 + \frac{4}{a^2}}}\left(e^{iT_0} + e^{-iT_0}\right)$

$$= \frac{2\cos T_0}{\sqrt{3T_1 + \frac{4}{a^2}}}$$

(3) $$\frac{d^2u}{dt^2} + u = \varepsilon\left[u^3 + 3\frac{du}{dt} - \left(\frac{du}{dt}\right)^3\right] \qquad u(0)=1 \quad \frac{du}{dt}(0)=0$$

As before; $T_0 = t$, $T_1 = \varepsilon t$, $u = u_0 + \varepsilon u_1$

$$\Rightarrow \left(\frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2}\right)(u_0 + \varepsilon u_1) + (u_0 + \varepsilon u_1)$$
$$= \varepsilon[\ldots]$$

→ To order zero:

$$\begin{cases} \dfrac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \\ u_0(0,0) = 1 \\ \dfrac{du_0}{dt}(0,0) = 0 \end{cases} \rightarrow u_0(T_0,T_1) = A_0(T_1)e^{iT_0} + A_0^*(T_1)e^{-iT_0}$$

with $\begin{cases} A_0 + A_0^*(0) = 1 \\ A_0 - A_0^*(0) = 0 \end{cases}$

$\rightarrow A_0 = A_0^*(0) = \frac{1}{2}$

To first order:

$$2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} + \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = u_0^3 + 3\frac{\partial u_0}{\partial T_0} - \left(\frac{\partial u_0}{\partial T_0}\right)^3$$

$$\Rightarrow \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2i\left(\frac{\partial A_0}{\partial T_1}e^{iT_0} - \frac{\partial A_0^*}{\partial T_1}e^{-iT_0}\right)$$
$$+ \left(A_0 e^{iT_0} + A_0^* e^{-iT_0}\right)^3$$
$$+ 3i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)$$
$$+ i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)^3$$

$$= -i\left(2\frac{\partial A_0}{\partial T_1}e^{iT_0} - 2\frac{\partial A_0^*}{\partial T_1}e^{-iT_0}\right)$$
$$+ \left(A_0^3 e^{3iT_0} + 3A_0^2 A_0^* e^{iT_0} + 3A_0 A_0^* e^{-iT_0} + A_0^{*3} e^{-3iT_0}\right)$$
$$+ 3i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)$$
$$+ i\left(A_0^3 e^{3iT_0} - 3A_0^2 A_0^* e^{iT_0} + 3A_0^* A_0 e^{-iT_0} - A_0^{*3} e^{-3iT_0}\right)$$

$\Rightarrow$ the compatibility condition is:

$$-i2\frac{\partial A_0}{\partial T_1} + 3A_0^2A_0^* + 3iA_0 - 3iA_0^2A_0^* = 0$$

$$\Rightarrow 2i\frac{\partial A_0}{\partial T_1} = 3(1-i)A_0^2A_0^* + 3iA_0$$

$$2\frac{\partial A_0}{\partial T_1} = 3(-i-1)A_0^2A_0^* + 3A_0$$

let $A_0 = |A_0|e^{i\theta}$ then

$$2\frac{d|A_0|}{dT_1}e^{i\theta} + 2i\frac{d\theta}{dT_1}|A_0|e^{i\theta} = -3(i+1)|A_0|^2|A_0|e^{i\theta} + 3|A_0|e^{i\theta}$$

$$\Rightarrow 2\frac{d|A_0|}{dT_1} + 2i\frac{d\theta}{dT_1}|A_0| = -3|A_0|^3 + 3|A_0| - 3i|A_0|^3$$

$$\Rightarrow \begin{cases} 2\frac{d|A_0|}{dT_1} = -3|A_0|(|A_0|^2-1) \\ 2\frac{d\theta}{dT_1} = -3|A_0|^2 \end{cases}$$

$\Rightarrow$ The $|A_0|$ equation:

$$\frac{d|A_0|}{|A_0|(1-|A_0|^2)} = \frac{3}{2}dT_1 \Rightarrow \ln\sqrt{\frac{|A_0|^2}{|1-|A_0|^2|}} = \frac{3}{2}T_1 + C$$

$|A_0| = \frac{1}{2}$ at $T_1 = 0$ so $C_1 = \ln\sqrt{\frac{1/4}{|1-1/4|}}$

$$= \ln\sqrt{\frac{1}{3}}$$

$$= -\frac{1}{2}\ln 3$$

$$\Rightarrow \quad \ln\sqrt{\frac{|A_0|^2}{|1-|A_0|^2|}} = \frac{3T_1}{2} - \frac{1}{2}\ln 3 \Rightarrow$$

$$\frac{|A_0|^2}{|1-|A_0|^2|} = e^{3T_1 - \ln 3} = \left(\frac{1}{3}\right)e^{3T_1}$$

$\Rightarrow$ if $|A_0| < 1$ then

$$|A_0|^2 = \left(1-|A_0|^2\right)\frac{1}{3}e^{3T_1}$$

$$\Rightarrow \quad |A_0|^2 = \frac{\frac{1}{3}e^{3T_1}}{1+\frac{1}{3}e^{3T_1}} = \frac{1}{3e^{-3T_1}+1}$$

$$\Rightarrow \quad |A_0| = \frac{1}{\sqrt{1+3e^{-3T_1}}}$$

and finally, we can attack the θ equation

$$2\frac{d\theta}{dT_1} = \frac{-3}{3e^{-3T_1}+1}$$

$$\Rightarrow \quad 2\frac{d\theta}{3} = -\frac{dT_1}{3e^{-3T_1}+1} = -dT_1\,\frac{\frac{1}{3}e^{3T_1}}{1+\frac{1}{3}e^{3T_1}}$$

$$= -\frac{dT_1}{3}\cdot\frac{1}{\cancel{2}}\,\frac{\cancel{2}e^{3T_1}}{1+\frac{1}{3}e^{3T_1}}$$

$$2\frac{d\theta}{3} = -\frac{dT_1}{3}\,\frac{e^{3T_1}}{1+\frac{1}{3}e^{6T_1}}$$

$$\Rightarrow \quad \theta = -\frac{1}{2}\ln\left(1+\frac{1}{3}e^{3T_1}\right) + \theta_0$$

at $T_1 = 0$, $\theta = 0$ so $\theta_0 = \frac{1}{2}\ln\left(\frac{4}{3}\right)$

$$\theta(T_1) = -\frac{1}{2}\ln\left(1+\frac{1}{3}e^{3T_1}\right) + \frac{1}{2}\ln\left(\frac{4}{3}\right) = \frac{-1}{2}\ln\left(\frac{3+e^{3T_1}}{4}\right)$$

$$\Rightarrow \quad A_0(T_1) = \frac{1}{\sqrt{1+3e^{-3T_1}}}\,e^{-\frac{i}{2}\ln\left(1+\frac{1}{3}e^{3T_1}\right)}$$

$$\Rightarrow \quad u(T_0,T_1) = \frac{1}{\sqrt{1+3e^{-3T_1}}}\cdot\left[e^{iT_0-\frac{i}{2}\ln\left(\frac{3+e^{3T_1}}{4}\right)}\right.$$

$$\left.+\,e^{-iT_0+\frac{i}{2}\ln\left(\frac{3+e^{3T_1}}{4}\right)}\right]$$

$$= \frac{2}{\sqrt{1+3e^{-3T_1}}}\cos\left(T_0-\frac{1}{2}\ln\left(\frac{3+e^{3T_1}}{4}\right)\right).$$

$$\rightarrow u(t) = \frac{2}{\sqrt{1+3e^{-3\varepsilon t}}} \cos\left(t - \frac{1}{2}\ln\left(\frac{3+e^{3\varepsilon t}}{4}\right)\right)$$

as t grows $\rightarrow \infty$

$$u(t) \rightarrow 2\cos\left(t - \frac{1}{2}\ln\left(\frac{1}{4}e^{3\varepsilon t}\right)\right)$$

$$= 2\cos\left(t + \frac{1}{2}\ln 2 - \frac{1}{2}\ln(e^{3\varepsilon t})\right)$$

$$= 2\cos\left(t + \frac{1}{2}\ln 2 - \frac{3}{2}\varepsilon t\right)$$

$$= 2\cos\left(\left(1 - \frac{3}{2}\varepsilon\right)t + \frac{1}{2}\ln 2\right)$$

↑ the perturbation causes a change of phase & period as $t \rightarrow \infty$.

(5) $\frac{d^2u}{dt^2} + u = \varepsilon\left(u^3 - u^2\frac{du}{dt} + 4\frac{du}{dt}\right) \quad u(0) = 2 \quad \frac{du}{dt}(0) = 0$

As usual

To zeroth order

$$\begin{cases} \frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \\ u_0(0,0) = 2 \\ \frac{\partial u_0}{\partial T_0} = 0 \end{cases} \rightarrow u_0(T_0, T_1) = A_0(T_1)e^{iT_0} + A_0^*(T_1)e^{-iT_0}$$

$$\begin{cases} A_0 + A_0^*(0) = 2 \\ A_0 - A_0^*(0) = 0 \end{cases}$$

$$\rightarrow A_0 = A_0^*(0) = 1$$

<u>To first order</u>

$$\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\left(\frac{\partial^2 u_0}{\partial T_1 \partial T_0}\right) + u_0^3 - u_0^2\frac{\partial u_0}{\partial T_0} + 4\frac{\partial u_0}{\partial T_0}$$

so $\frac{\partial^2 u_1}{\partial T_0^2} + u_1 =$

$$-2i\left(\frac{\partial A_0}{\partial T_1} e^{iT_0} - \frac{\partial A_0^*}{\partial T_1} e^{-iT_0}\right)$$
$$+\left(A_0 e^{iT_0} + A_0^* e^{-iT_0}\right)^3$$
$$-\left(A_0 e^{iT_0} + A_0^* e^{-iT_0}\right)^2 i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)$$
$$+4i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)$$

$$= -2i\left(\frac{\partial A_0}{\partial T_1} e^{iT_0} - \frac{\partial A_0^*}{\partial T_1} e^{-iT_0}\right)$$
$$+\left(A_0^3 e^{3iT_0} + 3A_0^2 A_0^* e^{iT_0} + 3A_0 A_0^{*2} e^{-iT_0} + A_0^{*3} e^{-3iT_0}\right)$$
$$-i\left(A_0^2 e^{2iT_0} + 2A_0 A_0^* + A_0^{*2} e^{-2iT_0}\right)\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)$$
$$+4i\left(A_0 e^{iT_0} - A_0^* e^{-iT_0}\right)$$

$\Rightarrow$ Compatibility condition:

$$-2i\frac{\partial A_0}{\partial T_1} + 3A_0^2 A_0^* + iA_0^2 A_0^* - 2iA_0^2 A_0^* + 4iA_0 = 0$$

$$\Rightarrow \quad -2i\frac{\partial A_0}{\partial T_1} + (3-i)A_0^2 A_0^* + 4iA_0 = 0$$

$$\Rightarrow \quad i\frac{\partial A_0}{\partial T_1} = \frac{1}{2}(3-i)A_0^2 A_0^* + 2iA_0$$

$$\frac{\partial A_0}{\partial T_1} = \frac{1}{2}(-3i-1)A_0^2 A_0^* + 2A_0$$

$\Rightarrow$ let $A_0 = |A_0|e^{i\theta}$ then $A_0^* = |A_0|e^{-i\theta}$

$$\frac{d|A_0|}{dT_1}e^{i\theta} + i\frac{d\theta}{dT_1}|A_0|e^{i\theta} = -\frac{1}{2}(3i+1)|A_0|^2|A_0|e^{i\theta} + 2|A_0|e^{i\theta}$$

$$\Rightarrow \quad \begin{cases} \dfrac{d|A_0|}{dT_1} = -\dfrac{1}{2}|A_0|^2|A_0| + 2|A_0| \\[2ex] \dfrac{d\theta}{dT_1} = -\dfrac{3}{2}|A_0|^2 \end{cases}$$

The $|A_0|$ equation has solution (see Wolfram a for instance

$$|A_0|(T_1) = \pm \frac{2e^{2T_1}}{\sqrt{K + e^{4T_1}}}$$

Since $A_0(0) = 1$, $\quad 1 = \frac{\pm 2}{\sqrt{K+1}} \Rightarrow$

$K+1 = 4 \Rightarrow K = 3$

also, it has to be + solution $\left(1 = \frac{2}{\sqrt{4}}\right)$

$$\Rightarrow |A_0|(T_1) = \frac{2e^{2T_1}}{\sqrt{3 + e^{4T_1}}}$$

$$\frac{d\theta}{dT_1} = -\frac{3}{2}|A_0|^2 = -\frac{3}{2}\,\frac{4e^{4T_1}}{3 + e^{4T_1}} = \frac{-6e^{4T_1}}{3 + e^{4T_1}}$$

Solve by separation of variables

$$\Rightarrow \frac{d\theta}{6} = -\frac{dT_1\, e^{4T_1}}{3 + e^{4T_1}} \Rightarrow$$

integrate both sides $\Rightarrow -\frac{1}{4}\ln\left(e^{4T_1} + 3\right) = \frac{\theta}{6}$ + const

$$\text{so} \quad \theta = -\frac{3}{2}\ln\left(e^{4T_1} + 3\right) + K$$

Since $A_0(0)$ is real, $\theta(0) = 0 \Rightarrow \quad K = \frac{3}{2}\ln 4$

$$\theta = -\frac{3}{2}\ln\left(e^{4T_1} + 3\right) + \frac{3}{2}\ln 4 = -\frac{3}{2}\ln\left(\frac{3 + e^{4T_1}}{4}\right)$$

$$\rightarrow \text{finally,} \quad A_0(T) = \frac{2e^{2T_1}}{\sqrt{3 + e^{4T_1}}}\, e^{-\frac{3i}{2}\ln\left(\frac{e^{4T_1}+3}{4}\right)}$$

$$u(T_0, T_1) = \frac{2e^{2T_1}}{\sqrt{3 + e^{4T_1}}}\left(e^{iT_0 - \frac{3}{2}i\ln\left(\frac{e^{4T_1}+3}{4}\right)} + e^{-iT_0 + \frac{3}{2}i\ln\left(\frac{e^{4T_1}+3}{4}\right)}\right)$$

$$= \frac{4e^{2T_1}}{\sqrt{3 + e^{4T_1}}}\cos\left(T_0 - \frac{3}{2}\ln\left(\frac{e^{4T_1}+3}{4}\right)\right)$$

$$\Rightarrow u(t) = \frac{4e^{2\varepsilon t}}{\sqrt{3 + e^{4\varepsilon t}}}\cos\left(t - \frac{3}{2}\ln\left(\frac{e^{4\varepsilon t}+3}{4}\right)\right) \rightarrow$$